On Bisimulations for Description Logics*

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Abstract. We formulate bisimulations for useful description logics. The simplest among the considered logics is a variant of PDL (propositional dynamic logic). The others extend that logic with inverse roles, nominals, quantified number restrictions, the universal role, and/or the concept constructor for expressing the local reflexivity of a role. They also allow role axioms. We give results about invariance of concepts, TBoxes and ABoxes, preservation of RBoxes and knowledge bases, and the Hennessy-Milner property w.r.t. bisimulations in the considered description logics. We also provide results on the largest auto-bisimulations and quotient interpretations w.r.t. such equivalence relations. Such results are useful for minimizing interpretations and concept learning in description logics. To deal with minimizing interpretations for the case when the considered logic allows quantified number restrictions and/or the constructor for the local reflexivity of a role, we introduce a new notion called QS-interpretation, which is needed for obtaining expected results. By adapting Hopcroft's automaton minimization algorithm, we give an efficient algorithm for computing the partition corresponding to the largest auto-bisimulation of a finite interpretation.

1 Introduction

Bisimulations arose in modal logic [20,21,22] and state transition systems [17,9]. They were introduced by van Benthem under the name *p-relation* in [20,21] and the name *zigzag relation* in [22]. Bisimulations reflect, in a particularly simple and direct way, the locality of the modal satisfaction definition. The famous Van Benthem Characterization Theorem states that modal logic is the bisimulation invariant fragment of first-order logic. Bisimulations have been used to analyze the expressivity of a wide range of extended modal logics (see, e.g., [2] for details). In state transition systems, bisimulation is viewed as a binary relation associating systems which behave in the same way in the sense that one system simulates the other and vice versa. Kripke models in modal logic are a special case of labeled state transition systems. Hennessy and Milner [9] showed that weak modal languages could be used to classify various notions of process invariance. In general, bisimulations are a very natural notion of equivalence for both mathematical and computational investigations.[1]

Bisimilarity between two states is usually defined by three conditions (the states have the same label, each transition from one of the states can be simulated by a similar transition from the other, and vice versa). As shown in [2], the four program constructors of PDL (propositional dynamic logic) are "safe" for these three conditions. That is, we need to specify the mentioned conditions only for atomic programs, and as a consequence, they hold also for complex programs. For bisimulation between two pointed-models, the initial states of the models are also required to be bisimilar. When converse is allowed (the case of CPDL), two additional conditions are required for bisimulation [2]. Bisimulation conditions for dealing with graded modalities were studied in [4,3,11]. In the field of hybrid logic, the bisimulation condition for dealing with nominals is well known (see, e.g., [1]).

Description logics (DLs) are variants of modal logic. They are of particular importance in providing a logical formalism for ontologies and the Semantic Web. DLs represent the

* This is an extended version of the workshop paper [6].

[1] This paragraph is based on [2].

domain of interest in terms of concepts, individuals, and roles. A concept is interpreted as a set of individuals, while a role is interpreted as a binary relation among individuals. A DL is characterized by a set of concept constructors, a set of role constructors, and a set of allowed forms of role axioms and individual assertions. A knowledge base in a DL usually has three parts: an RBox consisting of axioms about roles, a TBox consisting of terminology axioms, and an ABox consisting of assertions about individuals.

In this paper we study bisimulations for the family of DLs which extend $\mathcal{ALC}_{reg}$ (a variant of PDL) with an arbitrary combination of inverse roles, quantified number restrictions, nominals, the universal role, and the concept constructor $\exists r.\mathtt{Self}$ for expressing the local reflexivity of a role. Inverse roles are like converse modal operators, quantified number restrictions are like graded modalities, and nominals are as in hybrid logic.

The topic is worth studying due to the following reasons:

1. Despite that bisimulation conditions are known for PDL and for some features like converse modal operators, graded modal operators and nominals, we are not aware of previous work on bisimulation conditions for the universal role and the concept constructor $\exists r.\mathtt{Self}$. More importantly, without proofs one cannot be sure that all the conditions can be combined together to guarantee standard properties like invariance and the Hennessy-Milner property.

   There are many papers on bisimulations, but just a few on bisimulations in DLs:
   - In [12] Kurtonina and de Rijke studied expressiveness of concept expressions in some DLs by using bisimulations. They considered a family of DLs that are sublogics of the DL $\mathcal{ALCNR}$, which extend $\mathcal{ALC}$ with (unquantified) number restrictions and role conjunction. They did not consider individuals, nominals, quantified number restrictions, the concept constructor $\exists r.\mathtt{Self}$, the universal role, and the role constructors like the program constructors of PDL.
   - In [13] Lutz et al. characterized the expressiveness of TBoxes in the DL $\mathcal{ALCQIO}$ and its sublogics, including the lightweight DLs such as DL-Lite and $\mathcal{EL}$. They also studied invariance of TBoxes and the problem of TBox rewritability. The logic $\mathcal{ALCQIO}$ lacks the role constructors of PDL, the concept constructor $\exists r.\mathtt{Self}$ and the universal role.

   The family of DLs studied in this work is large and contains useful DLs. Not only concept constructors and role constructors are allowed, but role axioms are also allowed. In particular, the DL $\mathcal{SROIQ}$, which is the logical base of the Web Ontology Language OWL 2, belongs to this class.
2. DLs differ from other logics like modal logics and hybrid logics in the domain of applications and the settings. In DLs, there are special notions like named individual, RBox, TBox, ABox. Also, recall that a knowledge base in a DL usually consists of an RBox, a TBox and an ABox. Invariance of ABoxes and preservation of RBoxes and knowledge bases in DLs were not studied before. On the other hand, invariance of TBoxes was recently studied in the independent work [13] for the DL $\mathcal{ALCQIO}$ and its sublogics. Note that the first version [14] of [13] appeared to the public a few days later than the first version [7] of the current paper. The works [13,14] use the notion of global bisimulation to characterize invariance of TBoxes, whose condition is the same as the bisimulation conditions introduced in the current paper and [7] for the universal role.
3. Bisimulation is a very useful notion for DLs. Apart from analyzing expressiveness of DLs, in can be used for minimizing interpretations and concept learning in DLs:
   - Roughly speaking, two objects bisimilar to each other can be merged. This is the basis for minimizing interpretations. In automated reasoning in DLs, sometimes we want to return a model of a knowledge base (e.g., as a counter example for

a subsumption problem or an instance checking problem). It is expected that the returned model is simple and as small as possible. One can just find some model and minimize it. As another example, given an information system specified by an acyclic knowledge base with a large ABox and a small TBox, one can compute that information system and minimize it to save space and increase efficiency of reasoning tasks.

– Concept learning in DLs is similar to binary classification in traditional machine learning. The difference is that in DLs objects are described not only by attributes but also by relationship between the objects. As bisimulation is the notion for characterizing indiscernibility of objects in DLs, it is very useful for concept learning in DLs [15,19,8,5].

In this paper we present conditions for bisimulation in a uniform way for the whole considered family of DLs. A special point of our approach is that named individuals are treated as initial states, which requires an appropriate condition for bisimulation. As far as we know, bisimulation conditions for the universal role and the concept constructor $\exists r.\mathtt{Self}$ are first given by us. Our bisimulation condition for quantified number restrictions is simpler than the ones given for graded modalities in [4,3]. It is weaker than the one given for counting modalities in [11], but is strong enough to guarantee the Hennessy-Milner property for the class of finitely branching (image-finite) interpretations. We prove the standard invariance property (Theorem 3.4) and the Hennessy-Milner property (Theorem 4.1) and address the following problems:

– When is a TBox invariant for bisimulation? (Corollary 3.5 and Theorem 3.6)
– When is an ABox invariant for bisimulation? (Theorem 3.8)
– What can be said about preservation of RBoxes w.r.t. bisimulation? (Theorem 3.11)
– What can be said about invariance or preservation of knowledge bases w.r.t. bisimulation? (Theorems 3.13 and 3.14)

Furthermore, we give results (Theorems 5.3, 5.4, 5.5, 5.9 and 5.10) on the largest auto-bisimulation of an interpretation in a DL, the quotient interpretation w.r.t. that equivalence relation, and minimality of such a quotient interpretation. To deal with minimizing interpretations for the case when the considered logic allows quantified number restrictions and/or the concept constructor $\exists r.\mathtt{Self}$, we introduce a new notion called QS-interpretation, which is needed for obtaining expected results.

Computing the largest auto-bisimulations in modal logics and state transition systems is standard like Hopcroft's automaton minimization algorithm [10] and the Paige-Tarjan algorithm [16]. By adapting Hopcroft's automaton minimization algorithm, we give an efficient algorithm for computing the partition corresponding to the largest auto-bisimulation of a finite interpretation in any DL of the considered family. The adaptation involves the allowed constructors of the considered DLs.

The rest of this paper is structured as follows. In Section 2 we present notation and semantics of the DLs considered in this paper. In Section 3 we define bisimulations in those DLs and give our results on invariance and preservation w.r.t. such bisimulations. In Section 4 we give our results on the Hennessy-Milner property of the considered DLs. Section 5 is devoted to auto-bisimulation and minimization. Section 6 is devoted to computing the partition corresponding to the largest auto-bisimulation of a finite interpretation Section 7 concludes this work. All proofs of the results of this paper are presented in the appendix.

2 Notation and Semantics of Description Logics

Our languages use a finite set Σ_C of *concept names* (atomic concepts), a finite set Σ_R of *role names* (atomic roles), and a finite set Σ_I of *individual names*. Let $\Sigma = \Sigma_C \cup \Sigma_R \cup \Sigma_I$.

We denote concept names by letters like A and B, denote role names by letters like r and s, and denote individual names by letters like a and b.

We consider some (additional) *DL-features* denoted by I (*inverse*), O (*nominal*), Q (*quantified number restriction*), U (*universal role*), `Self`. A *set of DL-features* is a set consisting of some or zero of these names. We sometimes abbreviate sets of DL-features, writing e.g., OIQ instead of $\{O, I, Q\}$.

Let Φ be any set of DL-features and let $\mathcal{L}$ stand for $\mathcal{ALC}_{reg}$, which is the name of a DL corresponding to propositional dynamic logic (PDL). The DL language $\mathcal{L}_\Phi$ allows *roles* and *concepts* defined inductively as follows:

- if $r \in \Sigma_R$ then r is a role of $\mathcal{L}_\Phi$
- if $A \in \Sigma_C$ then A is a concept of $\mathcal{L}_\Phi$
- if R and S are roles of $\mathcal{L}_\Phi$ and C is a concept of $\mathcal{L}_\Phi$ then
  - ε, $R \circ S$, $R \sqcup S$, R^* and $C?$ are roles of $\mathcal{L}_\Phi$
  - $\top$, $\bot$, $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$ and $\exists R.C$ are concepts of $\mathcal{L}_\Phi$
  - if $I \in \Phi$ then R^- is a role of $\mathcal{L}_\Phi$
  - if $O \in \Phi$ and $a \in \Sigma_I$ then $\{a\}$ is a concept of $\mathcal{L}_\Phi$
  - if $Q \in \Phi$, $r \in \Sigma_R$ and n is a natural number then $\geq n\, r.C$ and $\leq n\, r.C$ are concepts of $\mathcal{L}_\Phi$
  - if $\{Q, I\} \subseteq \Phi$, $r \in \Sigma_R$ and n is a natural number then $\geq n\, r^-.C$ and $\leq n\, r^-.C$ are concepts of $\mathcal{L}_\Phi$
  - if $U \in \Phi$ then U is a role of $\mathcal{L}_\Phi$
  - if $\mathtt{Self} \in \Phi$ and $r \in \Sigma_R$ then $\exists r.\mathtt{Self}$ is a concept of $\mathcal{L}_\Phi$.

We use letters like R and S to denote arbitrary roles, and use letters like C and D to denote arbitrary concepts. A role stands for a binary relation, while a concept stands for a unary relation.

The intended meaning of the role constructors is the following:

- $R \circ S$ stands for the sequential composition of R and S
- $R \sqcup S$ stands for the set-theoretical union of R and S
- R^* stands for the reflexive and transitive closure of R
- $C?$ stands for the test operator (as of PDL)
- R^- stands for the *inverse* of R.

The concept constructors $\forall R.C$ and $\exists R.C$ correspond respectively to the modal operators $[R]C$ and $\langle R \rangle C$ of PDL. The concept constructors $\geq n\, R.C$ and $\leq n\, R.C$ are called *quantified number restrictions*. They correspond to graded modal operators.

An *interpretation* $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ consists of a non-empty set $\Delta^\mathcal{I}$, called the *domain* of $\mathcal{I}$, and a function $\cdot^\mathcal{I}$, called the *interpretation function* of $\mathcal{I}$, which maps every concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$, maps every role name r to a binary relation $r^\mathcal{I}$ on $\Delta^\mathcal{I}$, and maps every individual name a to an element $a^\mathcal{I}$ of $\Delta^\mathcal{I}$. The interpretation function $\cdot^\mathcal{I}$ is extended to complex roles and complex concepts as shown in Figure 1, where $\#\Gamma$ stands for the cardinality of the set Γ. We write $C^\mathcal{I}(x)$ to denote $x \in C^\mathcal{I}$, and write $R^\mathcal{I}(x, y)$ to denote $\langle x, y \rangle \in R^\mathcal{I}$.

We say that a role R is in the *converse normal form* (CNF) if the inverse constructor is applied in R only to role names. Since every role can be translated to an equivalent role in CNF,[2] in this paper we assume that roles are presented in the CNF.

We refer to elements of Σ_R also as *atomic roles*. Let $\Sigma_R^\pm = \Sigma_R \cup \{r^- \mid r \in \Sigma_R\}$. From now on, by *basic roles* we refer to elements of $\Sigma_R^\pm$ if the considered language allows inverse

[2] For example, $((r \sqcup s^-) \circ r^*)^- = (r^-)^* \circ (r^- \sqcup s)$.

$$
\begin{array}{rlrl}
(R \circ S)^{\mathcal{I}} &= R^{\mathcal{I}} \circ S^{\mathcal{I}} & \top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
(R \sqcup S)^{\mathcal{I}} &= R^{\mathcal{I}} \cup S^{\mathcal{I}} & \bot^{\mathcal{I}} &= \emptyset \\
(R^*)^{\mathcal{I}} &= (R^{\mathcal{I}})^* & (\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C?)^{\mathcal{I}} &= \{\langle x, x\rangle \mid C^{\mathcal{I}}(x)\} & (C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
\varepsilon^{\mathcal{I}} &= \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\} & (C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
U^{\mathcal{I}} &= \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} & \{a\}^{\mathcal{I}} &= \{a^{\mathcal{I}}\} \\
(R^-)^{\mathcal{I}} &= (R^{\mathcal{I}})^{-1} & (\exists r.\mathtt{Self})^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x, x)\}
\end{array}
$$

$$
\begin{aligned}
(\forall R.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y\, [R^{\mathcal{I}}(x, y) \text{ implies } C^{\mathcal{I}}(y)]\} \\
(\exists R.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \exists y\, [R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)] \\
(\geq n\, R.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \#\{y \mid R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)\} \geq n\} \\
(\leq n\, R.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \#\{y \mid R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)\} \leq n\}
\end{aligned}
$$

Fig. 1. Interpretation of complex roles and complex concepts.

roles, and refer to elements of Σ_R otherwise. In general, the language decides whether inverse roles are allowed in the considered context.

A *role (inclusion) axiom* in $\mathcal{L}_\Phi$ is an expression of the form $\varepsilon \sqsubseteq r$ or $R_1 \circ \ldots \circ R_k \sqsubseteq r$, where $k \geq 1$ and $R_1, \ldots, R_k$ are basic roles of $\mathcal{L}_\Phi$.[3] Given an interpretation $\mathcal{I}$, define that:

$$
\begin{aligned}
\mathcal{I} \models \varepsilon \sqsubseteq r &\ \text{ if } \ \varepsilon^{\mathcal{I}} \subseteq r^{\mathcal{I}} \\
\mathcal{I} \models R_1 \circ \ldots \circ R_k \sqsubseteq r &\ \text{ if } \ R_1^{\mathcal{I}} \circ \ldots \circ R_k^{\mathcal{I}} \subseteq r^{\mathcal{I}}
\end{aligned}
$$

We say that a role axiom φ is *valid* in $\mathcal{I}$ and $\mathcal{I}$ *validates* φ if $\mathcal{I} \models \varphi$. Note that reflexiveness and transitiveness of atomic roles are expressible by role axioms. When $I \in \Phi$ symmetry of an atomic role can also be expressed by a role axiom.

An *RBox* in $\mathcal{L}_\Phi$ is a finite set of role axioms in $\mathcal{L}_\Phi$. An interpretation $\mathcal{I}$ is a *model* of an RBox $\mathcal{R}$, denoted by $\mathcal{I} \models \mathcal{R}$, if it validates all the role axioms of $\mathcal{R}$.

A *terminological axiom* in $\mathcal{L}_\Phi$, also called a *general concept inclusion* (GCI) in $\mathcal{L}_\Phi$, is an expression of the form $C \sqsubseteq D$, where C and D are concepts in $\mathcal{L}_\Phi$. An interpretation $\mathcal{I}$ *validates* an axiom $C \sqsubseteq D$, denoted by $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

A *TBox* in $\mathcal{L}_\Phi$ is a finite set of terminological axioms in $\mathcal{L}_\Phi$. An interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$, if it validates all the axioms of $\mathcal{T}$.

An *individual assertion* in $\mathcal{L}_\Phi$ is an expression of one of the forms $C(a)$ (*concept assertion*), $R(a, b)$ (*positive role assertion*), $\neg R(a, b)$ (*negative role assertion*), $a = b$, and $a \neq b$, where C is a concept and R is a role in $\mathcal{L}_\Phi$.

Given an interpretation $\mathcal{I}$, define that:

$$
\begin{array}{ll}
\mathcal{I} \models a = b & \text{if } a^{\mathcal{I}} = b^{\mathcal{I}} \\
\mathcal{I} \models a \neq b & \text{if } a^{\mathcal{I}} \neq b^{\mathcal{I}} \\
\mathcal{I} \models C(a) & \text{if } C^{\mathcal{I}}(a^{\mathcal{I}}) \text{ holds} \\
\mathcal{I} \models R(a, b) & \text{if } R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \text{ holds} \\
\mathcal{I} \models \neg R(a, b) & \text{if } R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \text{ does not hold.}
\end{array}
$$

We say that $\mathcal{I}$ *satisfies* an individual assertion φ if $\mathcal{I} \models \varphi$.

An *ABox* in $\mathcal{L}_\Phi$ is a finite set of individual assertions in $\mathcal{L}_\Phi$. An interpretation $\mathcal{I}$ is a *model* of an ABox $\mathcal{A}$, denoted by $\mathcal{I} \models \mathcal{A}$, if it satisfies all the assertions of $\mathcal{A}$.

[3] This definition depends only on whether $\mathcal{L}_\Phi$ allows inverse roles, i.e., whether $I \in \Phi$.

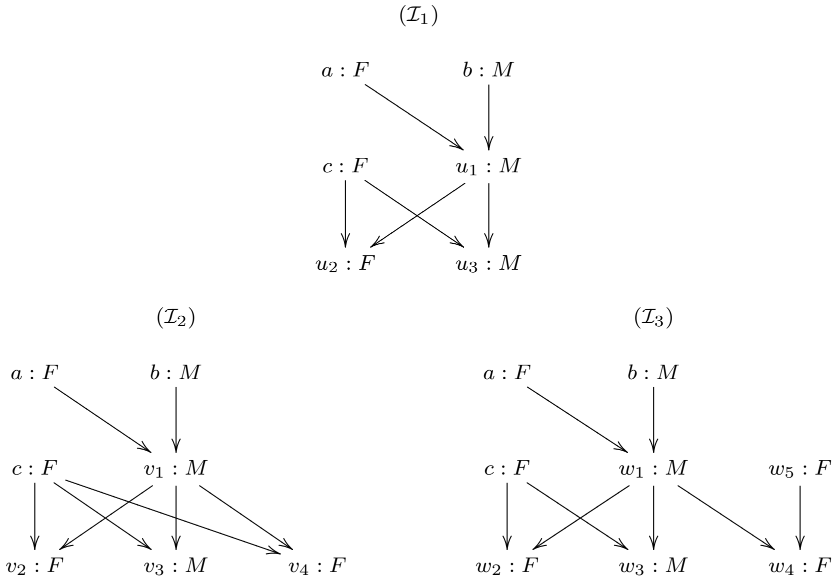


Fig. 2. Exemplary interpretations for Examples 2.1 and 3.2.

A *knowledge base* in $\mathcal{L}_\Phi$ is a triple $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{R}$ (resp. $\mathcal{T}$, $\mathcal{A}$) is an RBox (resp. a TBox, an ABox) in $\mathcal{L}_\Phi$. An interpretation $\mathcal{I}$ is a *model* of a knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ if it is a model of all $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{A}$.

Example 2.1. Let $\Sigma_I = \{a, b, c\}$, $\Sigma_C = \{F, M\}$ and $\Sigma_R = \{r\}$. One can think of these names as *Alice* (a), *Bob* (b), *Claudia* (c), *female* (F), *male* (M), and *has_child* (r). In Figure 2 we give three interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$. The edges are instances of r. We have, for example, $\Delta^{\mathcal{I}_1} = \{a^{\mathcal{I}_1}, b^{\mathcal{I}_1}, c^{\mathcal{I}_1}, u_1, u_2, u_3\}$, where these six elements are pairwise different, $F^{\mathcal{I}_1} = \{a^{\mathcal{I}_1}, c^{\mathcal{I}_1}, u_2\}$, and $M^{\mathcal{I}_1} = \{b^{\mathcal{I}_1}, u_1, u_3\}$.[4] All of these interpretations are models of the following ABox in $\mathcal{L}_{OIQ}$, where r^- can be read as *has_parent*:

$$\{\ F(a),\ M(b),\ F(c),\ (\exists r.(\exists r^-.\{b\} \sqcap {\geq}2\, r.\exists r^-.\{c\}))(a)\ \}$$

Assuming that r means *has_child*, then the last assertion of the above ABox means "a and b have a child which in turn has at least two children with c".

All the interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ validate the terminological axioms $\neg F \sqsubseteq M$ and $\{a\} \sqsubseteq \forall r^*.(\{a\} \sqcup {\geq}2\, r^-.\top)$ of $\mathcal{L}_{OIQ}$. $\lhd$

3 Bisimulations and Invariance Results

Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations. A binary relation $Z \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$ is called an $\mathcal{L}_\Phi$-*bisimulation* between $\mathcal{I}$ and $\mathcal{I}'$ if the following conditions hold for every $a \in \Sigma_I$, $A \in \Sigma_C$, $r \in \Sigma_R$, $x, y \in \Delta^{\mathcal{I}}$, $x', y' \in \Delta^{\mathcal{I}'}$:

$$Z(a^{\mathcal{I}}, a^{\mathcal{I}'}) \tag{1}$$

$$Z(x, x') \Rightarrow [A^{\mathcal{I}}(x) \Leftrightarrow A^{\mathcal{I}'}(x')] \tag{2}$$

$$[Z(x, x') \wedge r^{\mathcal{I}}(x, y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \wedge r^{\mathcal{I}'}(x', y')] \tag{3}$$

$$[Z(x, x') \wedge r^{\mathcal{I}'}(x', y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y, y') \wedge r^{\mathcal{I}}(x, y)], \tag{4}$$

[4] The elements u_i, v_j, w_k are unnamed objects. (The elements of Σ_I can be called *named individuals*, while the elements u_i, v_j, w_k can be called *unnamed individuals*.)

if $I \in \Phi$ then

$$[Z(x,x') \wedge r^{\mathcal{I}}(y,x)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y,y') \wedge r^{\mathcal{I}'}(y',x')] \tag{5}$$
$$[Z(x,x') \wedge r^{\mathcal{I}'}(y',x')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y,y') \wedge r^{\mathcal{I}}(y,x)], \tag{6}$$

if $O \in \Phi$ then

$$Z(x,x') \Rightarrow [x = a^{\mathcal{I}} \Leftrightarrow x' = a^{\mathcal{I}'}], \tag{7}$$

if $Q \in \Phi$ then

$$\begin{array}{l}\text{if } Z(x,x') \text{ holds then, for every role name } r \text{, there exists a bijection}\\ h : \{y \mid r^{\mathcal{I}}(x,y)\} \to \{y' \mid r^{\mathcal{I}'}(x',y')\} \text{ such that } h \subseteq Z,\end{array} \tag{8}$$

if $\{Q, I\} \subseteq \Phi$ then (additionally)

$$\begin{array}{l}\text{if } Z(x,x') \text{ holds then, for every role name } r \text{, there exists a bijection}\\ h : \{y \mid r^{\mathcal{I}}(y,x)\} \to \{y' \mid r^{\mathcal{I}'}(y',x')\} \text{ such that } h \subseteq Z,\end{array} \tag{9}$$

if $U \in \Phi$ then

$$\forall x \in \Delta^{\mathcal{I}}\, \exists x' \in \Delta^{\mathcal{I}'}\, Z(x,x') \tag{10}$$
$$\forall x' \in \Delta^{\mathcal{I}'}\, \exists x \in \Delta^{\mathcal{I}}\, Z(x,x'), \tag{11}$$

if $\mathtt{Self} \in \Phi$ then

$$Z(x,x') \Rightarrow [r^{\mathcal{I}}(x,x) \Leftrightarrow r^{\mathcal{I}'}(x',x')]. \tag{12}$$

For example, if $\Phi = \{Q, I\}$ then only the conditions (1)-(6), (8), (9) (and all of them) are essential.

Lemma 3.1.

1. *The relation $\{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}$.*
2. *If Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ then Z^{-1} is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}'$ and $\mathcal{I}$.*
3. *If Z_1 is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$, and Z_2 is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$, then $Z_1 \circ Z_2$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_2$.*
4. *If $\mathcal{Z}$ is a set of $\mathcal{L}_\Phi$-bisimulations between $\mathcal{I}$ and $\mathcal{I}'$ then $\bigcup \mathcal{Z}$ is also an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

The proof of this lemma is straightforward.

An interpretation $\mathcal{I}$ is $\mathcal{L}_\Phi$*-bisimilar* to $\mathcal{I}'$ if there exists an $\mathcal{L}_\Phi$-bisimulation between them. By Lemma 3.1, this $\mathcal{L}_\Phi$-bisimilarity relation is an equivalence relation between interpretations. We say that $x \in \Delta^{\mathcal{I}}$ is $\mathcal{L}_\Phi$*-bisimilar* to $x' \in \Delta^{\mathcal{I}'}$ if there exists an $\mathcal{L}_\Phi$-bisimulation Z between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x,x')$ holds. This latter $\mathcal{L}_\Phi$-bisimilarity relation is also an equivalence relation (between elements of interpretations' domains).

Example 3.2. Consider the interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ given in Figure 2 and described in Example 2.1. All of them are $\mathcal{L}$-bisimilar. The elements u_2 (of $\mathcal{I}_1$) and v_2, v_4 (of $\mathcal{I}_2$) are $\mathcal{L}_\Phi$-bisimilar for $\Phi \subseteq \{I, O\}$. The elements u_1 (of $\mathcal{I}_1$) and v_1 (of $\mathcal{I}_2$) are not $\mathcal{L}_Q$-bisimilar. The interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ are $\mathcal{L}_\Phi$-bisimilar for $\Phi \subseteq \{I, O\}$, but not $\mathcal{L}_Q$-bisimilar. The interpretation $\mathcal{I}_3$ is not $\mathcal{L}_I$-bisimilar to $\mathcal{I}_1$ and $\mathcal{I}_2$, but it is $\mathcal{L}_Q$-bisimilar to $\mathcal{I}_1$. ◁

Lemma 3.3. *Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations and Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then the following properties hold for every concept C in $\mathcal{L}_\Phi$, every role R in $\mathcal{L}_\Phi$, every $x, y \in \Delta^\mathcal{I}$, every $x', y' \in \Delta^{\mathcal{I}'}$, and every $a \in \mathcal{I}$:*

$$Z(x, x') \Rightarrow [C^\mathcal{I}(x) \Leftrightarrow C^{\mathcal{I}'}(x')] \tag{13}$$

$$[Z(x, x') \wedge R^\mathcal{I}(x, y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \wedge R^{\mathcal{I}'}(x', y')] \tag{14}$$

$$[Z(x, x') \wedge R^{\mathcal{I}'}(x', y')] \Rightarrow \exists y \in \Delta^\mathcal{I}[Z(y, y') \wedge R^\mathcal{I}(x, y)] \tag{15}$$

if $O \in \Phi$ then:

$$Z(x, x') \Rightarrow [R^\mathcal{I}(x, a^\mathcal{I}) \Leftrightarrow R^{\mathcal{I}'}(x', a^{\mathcal{I}'})]. \tag{16}$$

A concept C in $\mathcal{L}_\Phi$ is said to be *invariant for $\mathcal{L}_\Phi$-bisimulation* if, for any interpretations $\mathcal{I}, \mathcal{I}'$ and any $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$, if $Z(x, x')$ holds then $x \in C^\mathcal{I}$ iff $x' \in C^{\mathcal{I}'}$.

Theorem 3.4. *All concepts in $\mathcal{L}_\Phi$ are invariant for $\mathcal{L}_\Phi$-bisimulation.*

This theorem follows immediately from the assertion (13) of Lemma 3.3.

A TBox $\mathcal{T}$ in $\mathcal{L}_\Phi$ is said to be *invariant for $\mathcal{L}_\Phi$-bisimulation* if, for every interpretations $\mathcal{I}$ and $\mathcal{I}'$, if there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ then $\mathcal{I}$ is a model of $\mathcal{T}$ iff $\mathcal{I}'$ is a model of $\mathcal{T}$. The notions of whether an ABox or a knowledge base in $\mathcal{L}_\Phi$ is invariant for $\mathcal{L}_\Phi$-bisimulation are defined similarly.

Corollary 3.5. *If $U \in \Phi$ then all TBoxes in $\mathcal{L}_\Phi$ are invariant for $\mathcal{L}_\Phi$-bisimulation.*

An interpretation $\mathcal{I}$ is said to be *unreachable-objects-free* (w.r.t. the considered language) if every element of $\Delta^\mathcal{I}$ is reachable from some $a^\mathcal{I}$, where $a \in \Sigma_I$, via a path consisting of edges being instances of basic roles. An element of the domain of $\mathcal{I}$ is called a named object if it is $a^\mathcal{I}$ for some $a \in \Sigma_I$. From the point of view of users, named objects are the most important ones. Reachable unnamed objects (i.e., unnamed objects reachable from some named objects) are less important. Unreachable objects can be treated as redundant elements. For example, for the instance checking problem, unreachable objects do not affect the result. That is, the class of unreachable-objects-free interpretations is very natural.

Like Corollary 3.5, the following theorem concerns invariance of TBoxes w.r.t. $\mathcal{L}_\Phi$-bisimulation.

Theorem 3.6. *Let $\mathcal{T}$ be a TBox in $\mathcal{L}_\Phi$ and $\mathcal{I}, \mathcal{I}'$ be unreachable-objects-free interpretations (w.r.t. $\mathcal{L}_\Phi$) such that there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then $\mathcal{I}$ is a model of $\mathcal{T}$ iff $\mathcal{I}'$ is a model of $\mathcal{T}$.*

To justify that Corollary 3.5 and Theorem 3.6 are as strong as possible, we present here a simple example with $U \notin \Phi$ and one of $\mathcal{I}, \mathcal{I}'$ being not unreachable-objects-free such that $\mathcal{I}$ and $\mathcal{I}'$ are $\mathcal{L}_\Phi$-bisimilar but there exists a TBox $\mathcal{T}$ such that $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I}' \not\models \mathcal{T}$:

Example 3.7. Assume that $U \notin \Phi$ and let $\Sigma_C = \{A\}$, $\Sigma_R = \emptyset$, $\Sigma_I = \{a\}$ (i.e., the signature consists of only concept name A and individual name a). Let $\mathcal{I}$ and $\mathcal{I}'$ be the interpretations specified by: $\Delta^\mathcal{I} = \{a\}$, $\Delta^{\mathcal{I}'} = \{a, u\}$, $a^\mathcal{I} = a^{\mathcal{I}'} = a$, $A^\mathcal{I} = A^{\mathcal{I}'} = \{a\}$. It can be checked that $Z = \{\langle a, a\rangle\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. However, $\mathcal{I}$ is a model of the TBox $\{\top \sqsubseteq A\}$, while $\mathcal{I}'$ is not. ◁

As mentioned in the introduction, in the independent work [13] Lutz et al. use the notion of global bisimulation to characterize invariance of TBoxes, whose condition is the same as our bisimulation conditions (10) and (11) for the universal role. Their result on invariance of TBoxes is not stronger than our Corollary 3.5: one can just add U to Φ, and the

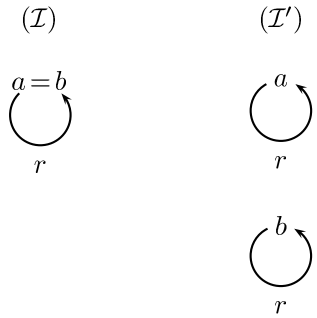
considered TBox, which may not use U, is invariant w.r.t. the corresponding bisimulation satisfying the conditions (10) and (11). On the matter of originality of our Corollary 3.5 and Theorem 3.6, as mentioned in the introduction, they appeared to the public in [7] a few days earlier than [14]. Besides, the family of DLs considered in the current paper contains other logics than the DL $\mathcal{ALCQIO}$ considered in [13].

The following theorem concerns invariance of ABoxes w.r.t. $\mathcal{L}_\Phi$-bisimulation.

Theorem 3.8. *Let $\mathcal{A}$ be an ABox in $\mathcal{L}_\Phi$. If $O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$ then $\mathcal{A}$ is invariant for $\mathcal{L}_\Phi$-bisimulation.*

Clearly, the condition "$O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$" of the above theorem covers many useful cases. The following example justifies that this theorem is as strong as possible.

Example 3.9. We show that if $O \notin \Phi$ then none of the ABoxes $\mathcal{A}_1 = \{a = b\}$, $\mathcal{A}_2 = \{a \neq b\}$, $\mathcal{A}_3 = \{r(a, b)\}$, $\mathcal{A}_4 = \{\neg r(a, b)\}$ is invariant for $\mathcal{L}_\Phi$-bisimulation. Assume that $O \notin \Phi$ and let $\Sigma_C = \emptyset$, $\Sigma_I = \{a, b\}$, $\Sigma_R = \{r\}$. Let $\mathcal{I}$ and $\mathcal{I}'$ be the interpretations specified by:

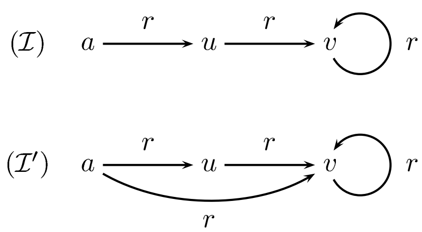


$\Delta^\mathcal{I} = \Delta^{\mathcal{I}'} = \{u, v\}$ with $u \neq v$, $a^\mathcal{I} = b^\mathcal{I} = a^{\mathcal{I}'} = u$, $b^{\mathcal{I}'} = v$, and $r^\mathcal{I} = r^{\mathcal{I}'} = \{\langle u, u\rangle, \langle v, v\rangle\}$. It can be checked that $Z = \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. However:

- $\mathcal{I}$ is a model of $\mathcal{A}_1$, while $\mathcal{I}'$ is not
- $\mathcal{I}'$ is a model of $\mathcal{A}_2$, while $\mathcal{I}$ is not
- $\mathcal{I}$ is a model of $\mathcal{A}_3$, while $\mathcal{I}'$ is not
- $\mathcal{I}'$ is a model of $\mathcal{A}_4$, while $\mathcal{I}$ is not. ◁

In general, RBoxes are not invariant for $\mathcal{L}_\Phi$-bisimulations. (The Van Benthem Characterization Theorem states that a first-order formula is invariant for bisimulations iff it is equivalent to the translation of a modal formula (see, e.g., [2]).) We give below a simple example about this:

Example 3.10. Let $\Sigma_C = \emptyset$, $\Sigma_R = \{r\}$, $\Sigma_I = \{a\}$ (i.e., the signature consists of only role name r and individual name a) and $\Phi = \emptyset$. Let $\mathcal{I}$ and $\mathcal{I}'$ be the interpretations specified by:



$\Delta^\mathcal{I} = \Delta^{\mathcal{I}'} = \{a, u, v\}$, $a^\mathcal{I} = a^{\mathcal{I}'} = a$, $r^\mathcal{I} = \{\langle a, u\rangle, \langle u, v\rangle, \langle v, v\rangle\}$ and $r^{\mathcal{I}'} = r^\mathcal{I} \cup \{\langle a, v\rangle\}$. It can be checked that $Z = \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. However, $\mathcal{I}'$ is a model of the RBox $\{r \circ r \sqsubseteq r\}$, while $\mathcal{I}$ is not. ◁

An interpretation $\mathcal{I}'$ is an r-extension of an interpretation $\mathcal{I}$ if $\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}}$, $\cdot^{\mathcal{I}'}$ differs from $\cdot^{\mathcal{I}}$ only in interpreting role names, and for all $r \in \Sigma_R$, $r^{\mathcal{I}'} \supseteq r^{\mathcal{I}}$.

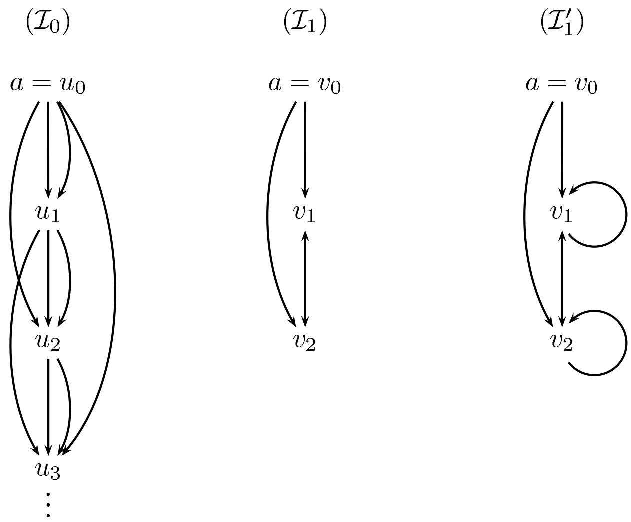
Given an interpretation $\mathcal{I}$ and an RBox $\mathcal{R}$, the *least r-extension of $\mathcal{I}$ validating $\mathcal{R}$* is the r-extension $\mathcal{I}'$ of $\mathcal{I}$ such that $\mathcal{I}'$ is a model of $\mathcal{R}$ and, for every r-extension $\mathcal{I}''$ of $\mathcal{I}$, if $\mathcal{I}''$ is a model of $\mathcal{R}$ then $r^{\mathcal{I}'} \subseteq r^{\mathcal{I}''}$ for all $r \in \Sigma_R$. That r-extension exists and is unique because the axioms of $\mathcal{R}$ correspond to non-negative Horn clauses of first-order logic.

Theorem 3.11. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{R}$ be an RBox in $\mathcal{L}_\Phi$. Let $\mathcal{I}_0$ be a model of $\mathcal{R}$, Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and an interpretation $\mathcal{I}_1$, and $\mathcal{I}_1'$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. Then Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1'$.*

This theorem states that, even in the case when interpretations $\mathcal{I}_0$ and $\mathcal{I}_1$ are $\mathcal{L}_\Phi$-bisimilar but $\mathcal{I}_0 \models \mathcal{R}$ while $\mathcal{I}_1 \not\models \mathcal{R}$, we can modify $\mathcal{I}_1$ slightly by adding some edges (i.e. instances of roles) to obtain a model $\mathcal{I}_1'$ of $\mathcal{R}$ that is $\mathcal{L}_\Phi$-bisimilar with $\mathcal{I}_0$ (and hence also with $\mathcal{I}_1$). This theorem is thus very natural.

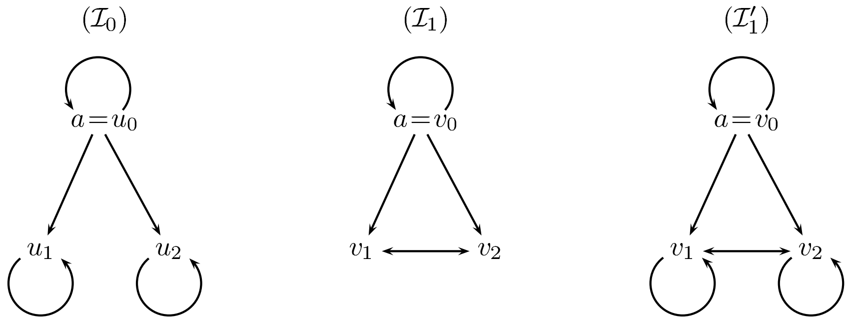
Example 3.12. To justify that the form of the above theorem is as strong as possible, we show that allowing either Q or $\mathtt{Self}$ in Φ can make the theorem wrong. In the following: $u_i \neq u_j$ if $i \neq j$; $v_i \neq v_j$ if $i \neq j$; and $u_i \neq v_j$ for all i, j. Here are examples:

1. Assume that $\mathtt{Self} \in \Phi$ and $\Phi \subseteq \{\mathtt{Self}, O, U\}$. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a\}$ and $\Sigma_R = \{r\}$. Let $\mathcal{I}_0$ and $\mathcal{I}_1$ be the interpretations specified by:



   - $\Delta^{\mathcal{I}_0} = \{u_i \mid i \geq 0\}$, $a^{\mathcal{I}_0} = u_0$, $r^{\mathcal{I}_0} = \{\langle u_i, u_j\rangle \mid i < j\}$
   - $\Delta^{\mathcal{I}_1} = \{v_0, v_1, v_2\}$, $a^{\mathcal{I}_1} = v_0$, $r^{\mathcal{I}_1} = \{\langle v_0, v_1\rangle, \langle v_1, v_2\rangle, \langle v_2, v_1\rangle, \langle v_0, v_2\rangle\}$.

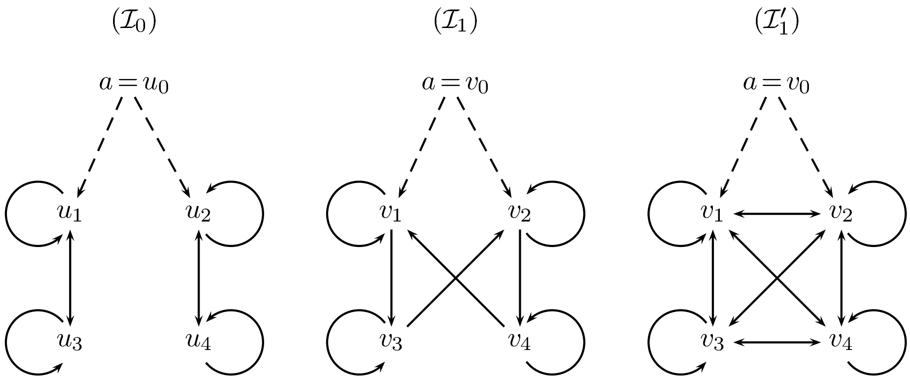
   Let $Z = \{\langle u_0, v_0\rangle\} \cup \{\langle u_i, v_j\rangle \mid i, j \geq 1\}$. It is easy to check that Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$, $\mathcal{I}_0$ is a model of the RBox $\mathcal{R} = \{r \circ r \sqsubseteq r\}$, but $\mathcal{I}_1$ is not. Let $\mathcal{I}_1'$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. We have that $\{\langle v_1, v_1\rangle, \langle v_2, v_2\rangle\} \subseteq r^{\mathcal{I}_1'}$, while $\langle u_i, u_i\rangle \notin r^{\mathcal{I}_0}$ for all $i \geq 0$. Hence $\{v_1, v_2\} \subseteq (\exists \mathtt{Self}.r)^{\mathcal{I}_1'}$, while $u_i \notin (\exists \mathtt{Self}.r)^{\mathcal{I}_0}$ for all $i \geq 0$, Thus, it is easy to check that Z is not an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1'$.
2. Assume that $Q \in \Phi$ and $\mathtt{Self} \notin \Phi$. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a\}$ and $\Sigma_R = \{r\}$. Let $\mathcal{I}_0$ and $\mathcal{I}_1$ be the interpretations specified by:



- $\Delta^{\mathcal{I}_0} = \{u_0, u_1, u_2\}$, $a^{\mathcal{I}_0} = u_0$,
  $r^{\mathcal{I}_0} = \{\langle u_0, u_0\rangle, \langle u_0, u_1\rangle, \langle u_0, u_2\rangle, \langle u_1, u_1\rangle, \langle u_2, u_2\rangle\}$
- $\Delta^{\mathcal{I}_1} = \{v_0, v_1, v_2\}$, $a^{\mathcal{I}_1} = v_0$,
  $r^{\mathcal{I}_1} = \{\langle v_0, v_0\rangle, \langle v_0, v_1\rangle, \langle v_0, v_2\rangle, \langle v_1, v_2\rangle, \langle v_2, v_1\rangle\}$.

Let $Z = \{\langle u_0, v_0\rangle\} \cup (\{u_1, u_2\} \times \{v_1, v_2\})$. It is easy to check that Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$, $\mathcal{I}_0$ is a model of the RBox $\mathcal{R} = \{\varepsilon \sqsubseteq r\}$, but $\mathcal{I}_1$ is not. Let $\mathcal{I}'_1$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. We have that $\{\langle v_1, v_1\rangle, \langle v_2, v_2\rangle\} \subseteq r^{\mathcal{I}'_1}$. Hence $\{v_1, v_2\} \subseteq (\geq 2\, r.\top)^{\mathcal{I}'_1}$, while $u_i \notin (\geq 2\, r.\top)^{\mathcal{I}_0}$ for both $i \in \{1, 2\}$. Thus, it is easy to check that Z is not an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}'_1$.

3. Assume that $Q \in \Phi$. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a\}$, $\Sigma_R = \{r, s\}$ and let $\mathcal{I}_0$, $\mathcal{I}_1$ be the interpretations specified by:



- $\Delta^{\mathcal{I}_0} = \{u_0, \ldots, u_4\}$, $a^{\mathcal{I}_0} = u_0$, $r^{\mathcal{I}_0} = \{\langle u_0, u_1\rangle, \langle u_0, u_2\rangle\}$,
  $s^{\mathcal{I}_0} = \{\langle u_i, u_j\rangle \mid \{i, j\} \subseteq \{1, 3\} \text{ or } \{i, j\} \subseteq \{2, 4\}\}$
- $\Delta^{\mathcal{I}_1} = \{v_0, \ldots, v_4\}$, $a^{\mathcal{I}_1} = v_0$, $r^{\mathcal{I}_1} = \{\langle v_0, v_1\rangle, \langle v_0, v_2\rangle\}$,
  $s^{\mathcal{I}_1} = \{\langle v_i, v_i\rangle \mid 1 \leq i \leq 4\} \cup \{\langle v_1, v_3\rangle, \langle v_3, v_2\rangle, \langle v_2, v_4\rangle, \langle v_4, v_1\rangle\}$.

Let $Z = \{\langle u_0, v_0\rangle\} \cup (\{u_1, u_2\} \times \{v_1, v_2\}) \cup (\{u_3, u_4\} \times \{v_3, v_4\})$. It is easy to check that Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$, $\mathcal{I}_0$ is a model of the RBox $\mathcal{R} = \{s \circ s \sqsubseteq s\}$, but $\mathcal{I}_1$ is not. Let $\mathcal{I}'_1$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. We have that $\{\langle v_3, v_4\rangle, \langle v_3, v_1\rangle\} \subseteq s^{\mathcal{I}'_1}$. Hence $v_3 \in (\geq 4\, s.\top)^{\mathcal{I}'_1}$, while $u_i \notin (\geq 4\, s.\top)^{\mathcal{I}_0}$ for all $0 \leq i \leq 4$. Thus, it is easy to check that Z is not an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}'_1$. $\triangleleft$

The following theorem concerns invariance of knowledge bases w.r.t. $\mathcal{L}_\Phi$-bisimulation. As stated before, in general, RBoxes are not invariant for $\mathcal{L}_\Phi$-bisimulations. Thus, it is natural to consider the case when the considered RBox is empty. Restricting to this case, generality of the below theorem follows from the generality of Theorems 3.6 and 3.8. The case when the considered RBox is not empty is addressed in Theorem 3.14.

Theorem 3.13. *Let $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ be a knowledge base in $\mathcal{L}_\Phi$ such that $\mathcal{R} = \emptyset$ and either $O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$. Let $\mathcal{I}$ and $\mathcal{I}'$ be unreachable-objects-free interpretations (w.r.t. $\mathcal{L}_\Phi$) such that there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then $\mathcal{I}$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ iff $\mathcal{I}'$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$.* $\triangleleft$

This theorem follows immediately from Theorems 3.6 and 3.8.

The following theorem concerns preservation of knowledge bases under $\mathcal{L}_\Phi$-bisimulation. Its generality follows from the generality of Theorems 3.6, 3.8 and 3.11. Clearly, it covers many useful cases.

Theorem 3.14. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ be a knowledge base in $\mathcal{L}_\Phi$ such that if $O \notin \Phi$ then $\mathcal{A}$ contains only assertions of the form $C(a)$. Let $\mathcal{I}_0$ and $\mathcal{I}_1$ be unreachable-objects-free interpretations (w.r.t. $\mathcal{L}_\Phi$) such that $\mathcal{I}_0$ is a model of $\mathcal{R}$ and there is an $\mathcal{L}_\Phi$-bisimulation Z between $\mathcal{I}_0$ and $\mathcal{I}_1$. Let $\mathcal{I}_1'$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. Then:*

1. *$\mathcal{I}_1'$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ iff $\mathcal{I}_0$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$*
2. *Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1'$.* $\lhd$

This theorem follows immediately from Theorems 3.6, 3.8 and 3.11.

4 The Hennessy-Milner Property

An interpretation $\mathcal{I}$ is *finitely branching* (or *image-finite*) w.r.t. $\mathcal{L}_\Phi$ if, for every $x \in \Delta^\mathcal{I}$ and every basic role R in $\mathcal{L}_\Phi$, the set $\{y \in \Delta^\mathcal{I} \mid R^\mathcal{I}(x, y)\}$ is finite.

Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations, and let $x \in \Delta^\mathcal{I}$ and $x' \in \Delta^{\mathcal{I}'}$. We say that x is $\mathcal{L}_\Phi$-*equivalent* to x' if, for every concept C in $\mathcal{L}_\Phi$, $x \in C^\mathcal{I}$ iff $x' \in C^{\mathcal{I}'}$.

Theorem 4.1 (The Hennessy-Milner Property). *Let $\mathcal{I}$ and $\mathcal{I}'$ be finitely branching interpretations (w.r.t. $\mathcal{L}_\Phi$) such that, for every $a \in \Sigma_I$, $a^\mathcal{I}$ is $\mathcal{L}_\Phi$-equivalent to $a^{\mathcal{I}'}$. Suppose that if $U \in \Phi$ then $\Sigma_I \neq \emptyset$. Then $x \in \Delta^\mathcal{I}$ is $\mathcal{L}_\Phi$-equivalent to $x' \in \Delta^{\mathcal{I}'}$ iff there exists an $\mathcal{L}_\Phi$-bisimulation Z between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x, x')$ holds. In particular, the relation $\{\langle x, x' \rangle \in \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'} \mid x$ is $\mathcal{L}_\Phi$-equivalent to $x'\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

Remark 4.2.

- Our Theorem 4.1 on the Hennessy-Milner property is formulated only for finitely branching interpretations. For larger classes of interpretations, one can use the notion of ω-saturatedness as in [13] to obtain a more general result. We did not investigate this yet. However, note that the class of finitely branching interpretations is very large and contains many interpretations of practical interest. For example, finite interpretations are finite branching.
- Our Theorem 4.1 presents *necessary* and *sufficient* conditions for invariance w.r.t. finitely branching interpretations.
- As mentioned earlier, our condition (8) for quantified number restrictions is simpler than the ones given for graded modalities in [4,3]. It is weaker than the one given for counting modalities in [11], but is strong enough to guarantee the Hennessy-Milner property for the class of finitely branching interpretations. It is necessary, sufficient and nice for that class of interpretations. $\lhd$

5 Auto-Bisimulation and Minimization

An $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and itself is called an $\mathcal{L}_\Phi$-*auto-bisimulation of* $\mathcal{I}$. An $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$ is said to be the *largest* if it is larger than or equal to ($\supseteq$) any other $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$.

Proposition 5.1. *For every interpretation $\mathcal{I}$, the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$ exists and is an equivalence relation.* $\lhd$

This proposition follows from Lemma 3.1.

Given an interpretation $\mathcal{I}$, by $\sim_{\Phi,\mathcal{I}}$ we denote the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$, and by $\equiv_{\Phi,\mathcal{I}}$ we denote the binary relation on $\Delta^\mathcal{I}$ with the property that $x \equiv_{\Phi,\mathcal{I}} x'$ iff x is $\mathcal{L}_\Phi$-equivalent to x'.

Theorem 5.2. *For every finitely branching interpretation $\mathcal{I}$, $\equiv_{\Phi,\mathcal{I}}$ is the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$ (i.e. the relations $\equiv_{\Phi,\mathcal{I}}$ and $\sim_{\Phi,\mathcal{I}}$ coincide).*

5.1 The Case without Q and `Self`

The *quotient interpretation* $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ of $\mathcal{I}$ w.r.t. $\sim_{\Phi,\mathcal{I}}$ is defined as usual:

- $\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = \{[x]_{\sim_{\Phi,\mathcal{I}}} \mid x \in \Delta^\mathcal{I}\}$, where $[x]_{\sim_{\Phi,\mathcal{I}}}$ is the abstract class of x w.r.t. $\sim_{\Phi,\mathcal{I}}$
- $a^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = [a^\mathcal{I}]_{\sim_{\Phi,\mathcal{I}}}$, for $a \in \Sigma_I$
- $A^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = \{[x]_{\sim_{\Phi,\mathcal{I}}} \mid x \in A^\mathcal{I}\}$, for $A \in \Sigma_C$
- $r^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = \{\langle [x]_{\sim_{\Phi,\mathcal{I}}}, [y]_{\sim_{\Phi,\mathcal{I}}}\rangle \mid \langle x, y\rangle \in r^\mathcal{I}\}$, for $r \in \Sigma_R$.

Theorem 5.3. *If $\Phi \subseteq \{I, O, U\}$ then, for every interpretation $\mathcal{I}$, the relation $Z = \{\langle x, [x]_{\sim_{\Phi,\mathcal{I}}}\rangle \mid x \in \Delta^\mathcal{I}\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$.*

The following theorem concerns invariance of terminological axioms and concept assertions, as well as preservation of role axioms and other individual assertion under the transformation of an interpretation to its quotient using the largest $\mathcal{L}_\Phi$-auto-bisimulation.

Theorem 5.4. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{I}$ be an interpretation. Then:*

1. *For every expression φ which is either a terminological axiom in $\mathcal{L}_\Phi$ or a concept assertion (of the form $C(a)$) in $\mathcal{L}_\Phi$, $\mathcal{I} \models \varphi$ iff $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*
2. *For every expression φ which is either a role inclusion axiom or an individual assertion of the form $R(a, b)$ or $a = b$, if $\mathcal{I} \models \varphi$ then $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*

An interpretation $\mathcal{I}$ is said to be *minimal* among a class of interpretations if $\mathcal{I}$ belongs to that class and, for every other interpretation $\mathcal{I}'$ of that class, $\#\Delta^\mathcal{I} \leq \#\Delta^{\mathcal{I}'}$ (the cardinality of $\Delta^\mathcal{I}$ is less than or equal to the cardinality of $\Delta^{\mathcal{I}'}$). The following theorem concerns minimality of quotient interpretations generated by using the largest $\mathcal{L}_\Phi$-auto-bisimulations.

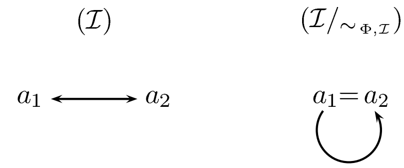
Theorem 5.5. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{I}$ be an unreachable-objects-free interpretation. Then:*

1. *$\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is a minimal interpretation $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$.*
2. *If $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finite then it is a minimal interpretation that validates the same terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*
3. *If $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finitely branching then it is a minimal interpretation that satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*

5.2 The Case with Q and/or `Self`

The following two examples show that we cannot make Theorems 5.3 and 5.4 stronger by allowing $\mathtt{Self} \in \Phi$ or $Q \in \Phi$.

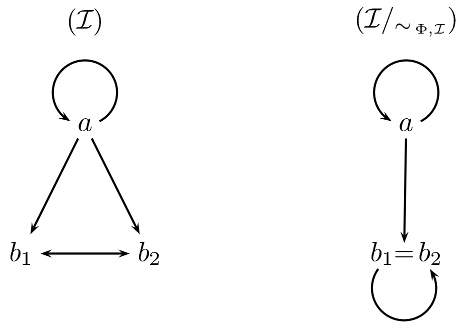
Example 5.6. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a_1, a_2\}$ and $\Sigma_R = \{r\}$, where $a_1 \neq a_2$. Consider the interpretation $\mathcal{I}$ specified by:



$\Delta^\mathcal{I} = \{a_1, a_2\}$, $a_1^\mathcal{I} = a_1$, $a_2^\mathcal{I} = a_2$ and $r^\mathcal{I} = \{\langle a_1, a_2\rangle, \langle a_2, a_1\rangle\}$. For any Φ, we have that $a_1 \sim_{\Phi,\mathcal{I}} a_2$. Denote $a = [a_1]_{\sim_{\Phi,\mathcal{I}}}$ $(= \{a_1, a_2\})$. The quotient interpretation $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is thus specified by: $\Delta^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = \{a\}$, $a_1^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = a_2^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = a$ and $r^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = \{\langle a, a\rangle\}$. Observe that if $\mathtt{Self} \in \Phi$ then:

- $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is not $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$,
- for φ being any of the axioms/assertions $\top \sqsubseteq \exists r.\mathtt{Self}$, $\varepsilon \sqsubseteq r$, $(\exists r.\mathtt{Self})(a_1)$, $a_1 = a_2$, $r(a_1, a_1)$, we have that $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$, but $\mathcal{I} \not\models \varphi$. $\triangleleft$

Example 5.7. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a, b_1, b_2\}$ and $\Sigma_R = \{r\}$, where a, b_1, b_2 are pairwise disjoint. Assume that $Q \in \Sigma$ and consider the interpretation $\mathcal{I}$ specified by:



$\Delta^\mathcal{I} = \{a, b_1, b_2\}$, $a^\mathcal{I} = a$, $b_1^\mathcal{I} = b_1$, $b_2^\mathcal{I} = b_2$ and $r^\mathcal{I} = \{\langle a, a\rangle, \langle a, b_1\rangle, \langle a, b_2\rangle, \langle b_1, b_2\rangle, \langle b_2, b_1\rangle\}$. Note that b_1 is $\mathcal{L}_\Phi$-bisimilar to b_2 and is not $\mathcal{L}_\Phi$-bisimilar to a. Denote $a' = [a]_{\sim_{\Phi,\mathcal{I}}}$ and $b' = [b_1]_{\sim_{\Phi,\mathcal{I}}}$ $(= \{b_1, b_2\})$. The quotient interpretation $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is thus specified by: $\Delta^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = \{a', b'\}$, $a^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = a'$, $b_1^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = b_2^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = b'$ and $r^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = \{\langle a', a'\rangle, \langle a', b'\rangle, \langle b', b'\rangle\}$. Observe that:

- $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is not $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$,
- for φ being any of the axioms/assertions $\geq 2\, r.\top \sqsubseteq\ \geq 3\, r.\top$, $\varepsilon \sqsubseteq r$, $(\geq 3\, r.\top)(a)$, $b_1 = b_2$, $r(b_1, b_1)$, we have that $\mathcal{I} \models \varphi$ iff $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \not\models \varphi$. $\triangleleft$

For the case when $Q \in \Phi$ or $\mathtt{Self} \in \Phi$, in order to obtain results similar to Theorems 5.4 and 5.5, we introduce QS-interpretations as follows.

A *QS-interpretation* is a tuple $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I}, \mathtt{Q}^\mathcal{I}, \mathtt{S}^\mathcal{I}\rangle$, where

- $\langle \Delta^\mathcal{I}, \cdot^\mathcal{I}\rangle$ is a traditional interpretation,
- $\mathtt{Q}^\mathcal{I}$ is a function that maps every basic role to a function $\Delta^\mathcal{I} \times \Delta^\mathcal{I} \to \mathbb{N}$ such that $\mathtt{Q}^\mathcal{I}(R)(x, y) > 0$ iff $\langle x, y\rangle \in R^\mathcal{I}$, where $\mathbb{N}$ is the set of natural numbers,
- $\mathtt{S}^\mathcal{I}$ is a function that maps every role name to a subset of $\Delta^\mathcal{I}$.

If $\mathcal{I}$ is a QS-interpretation then we redefine

$$(\exists r.\mathtt{Self})^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid x \in \mathtt{S}^\mathcal{I}(r)\}$$
$$(\geq n\, R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \Sigma\{\mathtt{Q}^\mathcal{I}(R)(x, y) \mid C^\mathcal{I}(y)\} \geq n\}$$
$$(\leq n\, R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \Sigma\{\mathtt{Q}^\mathcal{I}(R)(x, y) \mid C^\mathcal{I}(y)\} \leq n\}.$$

Other notions for interpretations remain unchanged for QS-interpretations.

For $\mathcal{I}$ being a traditional interpretation, the *quotient QS-interpretation* of $\mathcal{I}$ w.r.t. $\sim_{\Phi,\mathcal{I}}$, denoted by $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$, is the QS-interpretation $\mathcal{I}' = \langle \Delta^{\mathcal{I}'}, \cdot^{\mathcal{I}'}, \mathtt{Q}^{\mathcal{I}'}, \mathtt{S}^{\mathcal{I}'}\rangle$ such that:

- $\langle \Delta^{\mathcal{I}'}, \cdot^{\mathcal{I}'} \rangle$ is the quotient interpretation of $\mathcal{I}$ w.r.t. $\sim_{\Phi,\mathcal{I}}$
- for every basic role R and every $x, y \in \Delta^{\mathcal{I}}$,

$$\mathtt{Q}^{\mathcal{I}'}(R)([x]_{\sim_{\Phi,\mathcal{I}}}, [y]_{\sim_{\Phi,\mathcal{I}}}) = \max_{x' \in [x]_{\sim_{\Phi,\mathcal{I}}}} \#\{y' \in [y]_{\sim_{\Phi,\mathcal{I}}} \mid \langle x', y' \rangle \in R^{\mathcal{I}}\}$$

- for every role name r,

$$\mathtt{S}^{\mathcal{I}'}(r) = \{[x]_{\sim_{\Phi,\mathcal{I}}} \mid \langle x, x \rangle \in r^{\mathcal{I}}\}.$$

Note that, in the case when $Q \in \Phi$, we have

$$\mathtt{Q}^{\mathcal{I}'}(R)([x]_{\sim_{\Phi,\mathcal{I}}}, [y]_{\sim_{\Phi,\mathcal{I}}}) = \#\{y' \in [y]_{\sim_{\Phi,\mathcal{I}}} \mid \langle x, y' \rangle \in R^{\mathcal{I}}\}.$$

Lemma 5.8. *Let $\mathcal{I}$ be a traditional interpretation and let $\mathcal{I}' = \mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$. Then $Z = \{\langle x, [x]_{\sim_{\Phi,\mathcal{I}}} \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}\}$ satisfies all the properties (1)-(7), (10), (11), (13)-(16). In particular, the assertion (13) states that, for every concept C in $\mathcal{L}_\Phi$ and every $x \in \Delta^{\mathcal{I}}$, $x \in C^{\mathcal{I}}$ iff $[x]_{\sim_{\Phi,\mathcal{I}}} \in C^{\mathcal{I}'}$.*

The following theorem is a counterpart of Theorem 5.4, with no restrictions on Φ.

Theorem 5.9. *Let $\mathcal{I}$ be a traditional interpretation. Then:*

1. *For every expression φ which is either a terminological axiom in $\mathcal{L}_\Phi$ or a concept assertion (of the form $C(a)$) in $\mathcal{L}_\Phi$, $\mathcal{I} \models \varphi$ iff $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*
2. *For every expression φ which is either a role inclusion axiom or an individual assertion of the form $R(a, b)$ or $a = b$, if $\mathcal{I} \models \varphi$ then $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*

The following theorem is a counterpart of Theorem 5.5, with no restrictions on Φ.

Theorem 5.10. *Let $\mathcal{I}$ be a traditional interpretation without unreachable objects. Then:*

1. *If $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ is finite then it is a minimal QS-interpretation that validates the same terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*
2. *If $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ is finitely branching then it is a minimal QS-interpretation that satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*

6 Minimizing Interpretations

In this section, we adapt Hopcroft's automaton minimization algorithm [10] to computing the partition corresponding to $\sim_{\Phi,\mathcal{I}}$ for the case when $\mathcal{I}$ is finite. The partition is used to minimize $\mathcal{I}$ to obtain $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ for the case $\{Q, \mathtt{Self}\} \cap \Phi = \emptyset$, or $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ for the other case. We do not require any restrictions on Φ.

The similarity between minimizing automata and minimizing interpretations relies on that equivalence between two states in a finite deterministic automaton is similar to $\mathcal{L}_\Phi$-equivalence between two objects (i.e. elements of the domain) of an interpretation. The alphabet Σ of an automaton corresponds to Σ_R in the case $I \notin \Phi$, and corresponds to $\Sigma_R^{\pm}$ in the other case. There are two differences:

- In the case $Q \in \Phi$, objects x and x' in a block X should be separated by using a block Y and a basic role R (like a symbol of the alphabet) also in the case when the numbers of edges connecting x and x' to Y via R are different.
- The end conditions for equivalence are different: in the case of automata, it is required that the two considered states are either both accepting states or both unaccepting states; in the case of interpretations, it is required that the two considered objects x and x' satisfy the conjunction of the following conditions:

Algorithm 1: computing the partition corresponding to $\sim_{\Phi,\mathcal{I}}$

input : a set Φ of DL-features and a finite interpretation $\mathcal{I}$
output : the partition $\mathbb{P}$ corresponding to the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$

if $I \notin \Phi$ **then** let $\Sigma_R^\dagger = \Sigma_R$ **else** let $\Sigma_R^\dagger = \Sigma_R^\pm$;
set $\mathbb{P}$ to the partition corresponding to the equivalence relation $ECond_\Phi$;
set Z to a maximal block of $\mathbb{P}$;
set $\mathbb{L}$ to the empty collection;
foreach $X \in \mathbb{P} \setminus \{Z\}$ *and* $R \in \Sigma_R^\dagger$ **do** add $\langle X, R\rangle$ into $\mathbb{L}$;
while $\mathbb{L} \neq \emptyset$ **do**
  extract a pair $\langle Y, R\rangle$ from $\mathbb{L}$;
  foreach $X \in \mathbb{P}$ *split by* $\langle Y, R\rangle$ *(w.r.t.* Φ*)* **do**
    split X by $\langle Y, R\rangle$ (w.r.t. Φ) into a set $\mathbb{X}$ of blocks;
    set Z to a maximal block of $\mathbb{X}$;
    replace X in $\mathbb{P}$ by all the blocks of $\mathbb{X} \setminus \{Z\}$;
    foreach $S \in \Sigma_R^\dagger$ **do**
      if $\langle X, S\rangle \in \mathbb{L}$ **then** replace $\langle X, S\rangle$ in $\mathbb{L}$ by all the pairs $\langle X', S\rangle$ with $X' \in \mathbb{X}$
      else add all the pairs $\langle X', S\rangle$ with $X' \in \mathbb{X} \setminus \{Z\}$ into $\mathbb{L}$;

- for every $A \in \Sigma_C$, $x \in A^\mathcal{I}$ iff $x' \in A^\mathcal{I}$
- if $O \in \Phi$ then, for every $a \in \Sigma_I$, $x = a^\mathcal{I}$ iff $x' = a^\mathcal{I}$
- if $\mathtt{Self} \in \Phi$ then, for every $r \in \Sigma_R$, $\langle x, x\rangle \in r^\mathcal{I}$ iff $\langle x', x'\rangle \in r^\mathcal{I}$.

Denote the conjunction of the above three conditions by $ECond_\Phi(x, x')$.

Algorithm 1 (given on page 16) computes the partition corresponding to $\sim_{\Phi,\mathcal{I}}$ for the case when $\mathcal{I}$ is finite. It starts by partitioning $\Delta^\mathcal{I}$ into blocks using the equivalence relation $ECond_\Phi$ and after that follows the idea of Hopcroft's algorithm [10] to refine that partition. Like Hopcroft's algorithm, Algorithm 1 keeps the current partition $\mathbb{P}$ and a collection $\mathbb{L}$ of pairs $\langle Y, R\rangle$ for refining the partition, where $Y \in \mathbb{P}$ and R is a basic role. Splitting a block $X \in \mathbb{P}$ by a pair $\langle Y, R\rangle$ (w.r.t. Φ) is done according to the following principles:

1. If $Q \notin \Phi$ then $x, x' \in X$ are separated when only one among x and x' is connected to Y via R (i.e., $\#\{y \in Y \mid \langle x, y\rangle \in R^\mathcal{I}\} > 0$ iff $\#\{y \in Y \mid \langle x, y\rangle \in R^\mathcal{I}\} = 0$).
2. Otherwise, $x, x' \in X$ are separated when the numbers of edges connecting x and x' to Y via R are different (i.e., $\#\{y \in Y \mid \langle x, y\rangle \in R^\mathcal{I}\} \neq \#\{y \in Y \mid \langle x, y\rangle \in R^\mathcal{I}\}$). Note that X may be split into more than two blocks.

For the case $Q \notin \Phi$, refining the current partition using a pair $\langle Y, R\rangle$ is done in the same way as in Hopcroft's algorithm [10]. When $Q \in \Phi$, such a refinement is done analogously and by counting the number of edges connecting $x \in X$ to Y via R (i.e., instead of putting x into a subblock labeled by "$count = 0$" or "$count > 0$", we put it into a subblock labeled by "$count = k$" for an appropriate natural number k).

Theorem 6.1. *Algorithm 1 is correct and can be implemented to have time complexity* $O(|\Sigma| \cdot n \cdot \log n)$*, where* $n = |\Delta^\mathcal{I}|$.

7 Conclusions

We have studied bisimulations in a uniform way for a large class of DLs with useful ones like the DL $\mathcal{SROIQ}$ of OWL 2. In comparison with [12,13], this class allows also the role constructors of PDL, the concept constructor $\exists r.\mathtt{Self}$ and the universal role as well as role axioms. Our main contributions are the following:

- We proposed to treat named individuals as initial states and gave an appropriate condition for bisimulation. We are the first who gave bisimulation conditions for the universal role and the concept constructor $\exists r.\mathtt{Self}$.
- We proved that all of the bisimulation conditions (1)-(12) can be combined together to guarantee invariance of concepts and the Hennessy-Milner property for the whole class of studied DLs.
- We are the first one who addressed and gave results on invariance or preservation of ABoxes, RBoxes and knowledge bases in DLs. Independently and concurrently with [13] we gave first results on invariance of TBoxes. By examples, we showed that our results on invariance or preservation of TBoxes, ABoxes, RBoxes and knowledge bases in DLs are strong and cannot be extended in a straightforward way.
- We introduced a new notion called QS-interpretation, which is needed for dealing with minimizing interpretations in DLs with quantified number restrictions and/or the concept constructor $\exists r.\mathtt{Self}$.
- We formulated and proved results on minimality of quotient interpretations w.r.t. the largest auto-bisimulations.
- We adapted Hopcroft's automaton minimization algorithm to give an efficient algorithm for computing the partition corresponding to the largest auto-bisimulation of a finite interpretation in any DL of the considered family. The adaptation requires special treatments for the allowed constructors of the considered DLs.

This paper is a reasonably systematic work on bisimulations for DLs. What is missing in this paper is that we did not investigate expressiveness of DLs and did not generalize the Hennessy-Milner property for the class of non-finitely-branching interpretations. However, recall that: bisimulations can be used not only for analyzing expressiveness of logics but also for minimizing interpretations and concept learning in DLs; the class of finitely branching interpretations is very large and contains many interpretations of practical interest.

Our results found the logical base for concept learning in DLs [15,19,8,5]. These cited papers are pioneering ones in applying bisimulation to concept learning and approximation in DLs. That is, our results, especially the ones on the largest auto-bisimulations, are very useful for machine learning in the context of DLs.

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A Proofs

In this appendix we present proofs for the results of this paper. To increase readability, we recall our lemmas and theorems before presenting their proofs.

Lemma 3.3. *Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations and Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then the following properties hold for every concept C in $\mathcal{L}_\Phi$, every role R in $\mathcal{L}_\Phi$, every $x, y \in \Delta^\mathcal{I}$, every $x', y' \in \Delta^{\mathcal{I}'}$, and every $a \in \mathcal{I}$:*

$$(13) \quad Z(x,x') \Rightarrow [C^\mathcal{I}(x) \Leftrightarrow C^{\mathcal{I}'}(x')]$$
$$(14) \quad [Z(x,x') \wedge R^\mathcal{I}(x,y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y,y') \wedge R^{\mathcal{I}'}(x',y')]$$
$$(15) \quad [Z(x,x') \wedge R^{\mathcal{I}'}(x',y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y,y') \wedge R^{\mathcal{I}}(x,y)]$$

if $O \in \Phi$ then:

$$(16) \quad Z(x,x') \Rightarrow [R^\mathcal{I}(x,a^\mathcal{I}) \Leftrightarrow R^{\mathcal{I}'}(x',a^{\mathcal{I}'})].$$

Proof. We prove this lemma by induction on the structures of C and R.

Consider the assertion (14). Suppose $Z(x,x')$ and $R^\mathcal{I}(x,y)$ hold. By induction on the structure of R we prove that there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y,y')$ and $R^{\mathcal{I}'}(x',y')$ hold. The base case occurs when R is a role name and the assertion for it follows from (3). The induction steps are given below.

- Case $R = S_1 \circ S_2$: We have that $(S_1 \circ S_2)^\mathcal{I}(x,y)$ holds. Hence, there exists $z \in \Delta^\mathcal{I}$ such that $S_1^\mathcal{I}(x,z)$ and $S_2^\mathcal{I}(z,y)$ hold. Since $Z(x,x')$ and $S_1^\mathcal{I}(x,z)$ hold, by the inductive assumption of (14), there exists $z' \in \Delta^{\mathcal{I}'}$ such that $Z(z,z')$ and $S_1^{\mathcal{I}'}(x',z')$ hold. Since $Z(z,z')$ and $S_2^\mathcal{I}(z,y)$ hold, by the inductive assumption of (14), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y,y')$ and $S_2^{\mathcal{I}'}(z',y')$ hold. Since $S_1^{\mathcal{I}'}(x',z')$ and $S_2^{\mathcal{I}'}(z',y')$ hold, we have that $(S_1 \circ S_2)^{\mathcal{I}'}(x',y')$ holds, i.e. $R^{\mathcal{I}'}(x',y')$ holds.
- Case $R = S_1 \sqcup S_2$ is trivial.
- Case $R = S^*$: Since $R^\mathcal{I}(x,y)$ holds, there exist $x_0, \ldots, x_k \in \Delta^\mathcal{I}$ with $k \geq 0$ such that $x_0 = x$, $x_k = y$ and, for $1 \leq i \leq k$, $S^\mathcal{I}(x_{i-1},x_i)$ holds. Let $x'_0 = x'$. For each $1 \leq i \leq k$, since $Z(x_{i-1},x'_{i-1})$ and $S^\mathcal{I}(x_{i-1},x_i)$ hold, by the inductive assumption of (14), there exists $x'_i \in \Delta^{\mathcal{I}'}$ such that $Z(x_i,x'_i)$ and $S^{\mathcal{I}'}(x'_{i-1},x'_i)$ hold. Hence, $Z(x_k,x'_k)$ and $(S^*)^{\mathcal{I}'}(x'_0,x'_k)$ hold. Let $y' = x'_k$. Thus, $Z(y,y')$ and $R^{\mathcal{I}'}(x',y')$ hold.
- Case $R = (D?)$: Since $R^\mathcal{I}(x,y)$ holds, we have that $D^\mathcal{I}(x)$ holds and $x = y$. Since $Z(x,x')$ and $D^\mathcal{I}(x)$ hold, by the inductive assumption of (13), $D^{\mathcal{I}'}(x')$ also holds, and hence $R^{\mathcal{I}'}(x',x')$ holds. By choosing $y' = x'$, both $Z(y,y')$ and $R^{\mathcal{I}'}(x',y')$ hold.
- Case $I \in \Phi$ and $R = r^-$: The assertion for this case follows from (5).

By Lemma 3.1(2), the assertion (15) follows from the assertion (14).

Consider the assertion (16) and suppose $O \in \Phi$. By Lemma 3.1(2), it suffices to show that if $Z(x,x')$ and $R^\mathcal{I}(x,a^\mathcal{I})$ hold then $R^{\mathcal{I}'}(x',a^{\mathcal{I}'})$ also holds. We prove this by using similar argumentation as for (14). Suppose $Z(x,x')$ and $R^\mathcal{I}(x,a^\mathcal{I})$ hold. We prove that $R^{\mathcal{I}'}(x',a^{\mathcal{I}'})$ also holds by induction on the structure of R. The base case occurs when R is a role name and the assertion for it follows from (3) and (7). The induction steps are given below.

- Case $R = S_1 \circ S_2$: We have that $(S_1 \circ S_2)^\mathcal{I}(x,a^\mathcal{I})$ holds. Hence, there exists $y \in \Delta^\mathcal{I}$ such that $S_1^\mathcal{I}(x,y)$ and $S_2^\mathcal{I}(y,a^\mathcal{I})$ hold. Since $Z(x,x')$ and $S_1^\mathcal{I}(x,y)$ hold, by the inductive assumption of (14), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y,y')$ and $S_1^{\mathcal{I}'}(x',y')$ hold. Since $Z(y,y')$ and $S_2^\mathcal{I}(y,a^\mathcal{I})$ hold, by the inductive assumption of (16), $S_2^{\mathcal{I}'}(y',a^{\mathcal{I}'})$ holds. Since $S_1^{\mathcal{I}'}(x',y')$ and $S_2^{\mathcal{I}'}(y',a^{\mathcal{I}'})$ hold, we have that $(S_1 \circ S_2)^{\mathcal{I}'}(x',a^{\mathcal{I}'})$ holds, i.e. $R^{\mathcal{I}'}(x',a^{\mathcal{I}'})$ holds.

- Case $R = S_1 \sqcup S_2$ is trivial.
- Case $R = S^*$: Since $R^{\mathcal{I}}(x, a^{\mathcal{I}})$ holds, there exist $x_0, \ldots, x_k \in \Delta^{\mathcal{I}}$ with $k \geq 0$ such that $x_0 = x$, $x_k = a^{\mathcal{I}}$ and, for $1 \leq i \leq k$, $S^{\mathcal{I}}(x_{i-1}, x_i)$ holds.
  - Case $k = 0$: We have that $x = a^{\mathcal{I}}$. Since $Z(x, x')$ holds, by (7), it follows that $x' = a^{\mathcal{I}'}$. Hence $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ holds.
  - Case $k > 0$: Let $x'_0 = x'$. For each $1 \leq i < k$, since $Z(x_{i-1}, x'_{i-1})$ and $S^{\mathcal{I}}(x_{i-1}, x_i)$ hold, by the inductive assumption of (14), there exists $x'_i \in \Delta^{\mathcal{I}'}$ such that $Z(x_i, x'_i)$ and $S^{\mathcal{I}'}(x'_{i-1}, x'_i)$ hold. Hence, $Z(x_{k-1}, x'_{k-1})$ and $(S^*)^{\mathcal{I}'}(x'_0, x'_{k-1})$ hold. Since $Z(x_{k-1}, x'_{k-1})$ and $S^{\mathcal{I}}(x_{k-1}, a^{\mathcal{I}})$ hold, by the inductive assumption of (16), we have that $S^{\mathcal{I}'}(x'_{k-1}, a^{\mathcal{I}'})$ holds. Since $(S^*)^{\mathcal{I}'}(x'_0, x'_{k-1})$ holds, it follows that $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ holds.
- Case $R = (D?)$: Since $R^{\mathcal{I}}(x, a^{\mathcal{I}})$ holds, we have that $x = a^{\mathcal{I}}$ and $D^{\mathcal{I}}(a^{\mathcal{I}})$ holds. Since $Z(x, x')$ holds, by (7), it follows that $x' = a^{\mathcal{I}'}$. Since $Z(a^{\mathcal{I}}, a^{\mathcal{I}'})$ and $D^{\mathcal{I}}(a^{\mathcal{I}})$ hold, by the inductive assumption of (13), $D^{\mathcal{I}'}(a^{\mathcal{I}'})$ also holds. Since $x' = a^{\mathcal{I}'}$, it follows that $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ holds.
- Case $I \in \Phi$ and $R = r^-$: The assertion for this case follows from (5) and (7).

Consider the assertion (13). By Lemma 3.1(2), it suffices to show that if $Z(x, x')$ and $C^{\mathcal{I}}(x)$ hold then $C^{\mathcal{I}'}(x')$ also holds. Suppose $Z(x, x')$ and $C^{\mathcal{I}}(x)$ hold. The cases when C is of the form $\top$, $\bot$, A, $\neg D$, $D \sqcup D'$ or $D \sqcap D'$ are trivial.

- Case $C = \exists R.D$: Since $C^{\mathcal{I}}(x)$ holds, there exists $y \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y)$ and $D^{\mathcal{I}}(y)$ hold. Since $Z(x, x')$ and $R^{\mathcal{I}}(x, y)$ hold, by the assertion (14) (proved earlier), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold. Since $Z(y, y')$ and $D^{\mathcal{I}}(y)$ hold, by the inductive assumption of (13), it follows that $D^{\mathcal{I}'}(y')$ holds. Therefore, $C^{\mathcal{I}'}(x')$ holds.
- Case $C = \forall R.D$ is reduced to the above case, treating $\forall R.D$ as $\neg\exists R.\neg D$.
- Case $O \in \Phi$ and $C = \{a\}$: Since $C^{\mathcal{I}}(x)$ holds, we have that $x = a^{\mathcal{I}}$. Since $Z(x, x')$ holds, by (7), it follows that $x' = a^{\mathcal{I}'}$. Hence $C^{\mathcal{I}'}(x')$ holds.
- Case $Q \in \Phi$ and $C = (\geq n\, R.D)$, where R is a basic role: Since $Z(x, x')$ holds, there exists a bijection $h : \{y \mid R^{\mathcal{I}}(x, y)\} \to \{y' \mid R^{\mathcal{I}'}(x', y')\}$ such that $h \subseteq Z$. Since $C^{\mathcal{I}}(x)$ holds, there exist pairwise different $y_1, \ldots, y_n \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y_i)$ and $D^{\mathcal{I}}(y_i)$ hold for all $1 \leq i \leq n$. Consider any $1 \leq i \leq n$. Let $y'_i = h(y_i)$. Since $h \subseteq Z$, $Z(y_i, y'_i)$ must hold. Since $D^{\mathcal{I}}(y_i)$ holds, by the inductive assumption of (13), it follows that $D^{\mathcal{I}'}(y'_i)$ holds. Since $R^{\mathcal{I}'}(x', y'_i)$ and $D^{\mathcal{I}'}(y'_i)$ hold for all $1 \leq i \leq n$, it follows that $C^{\mathcal{I}'}(x')$ holds.
- Case $Q \in \Phi$ and $C = (\leq n\, R.D)$, where R is a basic role: This case is reduced to the above case, treating $\leq n\, R.D$ as $\neg(\geq (n+1)\, R.D)$.
- Case $\mathtt{Self} \in \Phi$ and $C = \exists r.\mathtt{Self}$: Since $C^{\mathcal{I}}(x)$ holds, we have that $r^{\mathcal{I}}(x, x)$ holds. By (12), it follows that $r^{\mathcal{I}'}(x', x')$ holds. Hence $C^{\mathcal{I}'}(x')$ holds. $\triangleleft$

Corollary 3.5. *If $U \in \Phi$ then all TBoxes in $\mathcal{L}_\Phi$ are invariant for $\mathcal{L}_\Phi$-bisimulation.*

Proof. Suppose $U \in \Phi$ and let $\mathcal{T}$ be a TBox in $\mathcal{L}_\Phi$ and $\mathcal{I}$, $\mathcal{I}'$ be interpretations. Suppose that $\mathcal{I}$ is a model of $\mathcal{T}$, and Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. We show that $\mathcal{I}'$ is a model of $\mathcal{T}$. Let $C \sqsubseteq D$ be an axiom from $\mathcal{T}$ and let $x' \in \Delta^{\mathcal{I}'}$. We need to show that $x' \in (\neg C \sqcup D)^{\mathcal{I}'}$. By (11), there exists $x \in \Delta^{\mathcal{I}}$ such that $Z(x, x')$ holds. Since $\mathcal{I}$ is a model of $\mathcal{T}$, we have that $x \in (\neg C \sqcup D)^{\mathcal{I}}$, which, by Theorem 3.4, implies that $x' \in (\neg C \sqcup D)^{\mathcal{I}'}$. $\triangleleft$

Theorem 3.6. *Let $\mathcal{T}$ be a TBox in $\mathcal{L}_\Phi$ and $\mathcal{I}$, $\mathcal{I}'$ be unreachable-objects-free interpretations (w.r.t. $\mathcal{L}_\Phi$) such that there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then $\mathcal{I}$ is a model of $\mathcal{T}$ iff $\mathcal{I}'$ is a model of $\mathcal{T}$.*

Proof. Let Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. By Lemma 3.1(2), it suffices to show that if $\mathcal{I}$ is a model of $\mathcal{T}$ then $\mathcal{I}'$ is also a model of $\mathcal{T}$. Suppose $\mathcal{I}$ is a model of $\mathcal{T}$. Let $C \sqsubseteq D$ be an axiom from $\mathcal{T}$. We need to show that $C^{\mathcal{I}'} \subseteq D^{\mathcal{I}'}$. Let $x' \in C^{\mathcal{I}'}$. We show that $x' \in D^{\mathcal{I}'}$.

Since $\mathcal{I}'$ is an unreachable-objects-free interpretation, there exist elements $x'_0, \ldots, x'_k$ of $\Delta^{\mathcal{I}'}$ and basic roles $R_1, \ldots, R_k$ with $k \geq 0$ such that $x'_0 = a^{\mathcal{I}'}$ for some $a \in \Sigma_I$, $x'_k = x'$ and, for $1 \leq i \leq k$, $R_i^{\mathcal{I}'}(x'_{i-1}, x'_i)$ holds.

By (1), $Z(a^{\mathcal{I}}, a^{\mathcal{I}'})$ holds. Let $x_0 = a^{\mathcal{I}}$. For each $1 \leq i \leq k$, since $Z(x_{i-1}, x'_{i-1})$ and $R_i^{\mathcal{I}'}(x'_{i-1}, x'_i)$ hold, by (15), there exists $x_i \in \Delta^{\mathcal{I}}$ such that $Z(x_i, x'_i)$ and $R_i^{\mathcal{I}}(x_{i-1}, x_i)$ hold. Let $x = x_k$. Thus, $Z(x, x')$ holds. Since $x' \in C^{\mathcal{I}'}$, by Theorem 3.4, we have that $x \in C^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{T}$, it follows that $x \in D^{\mathcal{I}}$. By Theorem 3.4, we derive that $x' \in D^{\mathcal{I}'}$, which completes the proof. ◁

Theorem 3.8. *Let $\mathcal{A}$ be an ABox in $\mathcal{L}_\Phi$. If $O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$ then $\mathcal{A}$ is invariant for $\mathcal{L}_\Phi$-bisimulation.*

Proof. Suppose that $O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$. Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations and let Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. By Lemma 3.1(2), it suffices to show that if $\mathcal{I}$ is a model of $\mathcal{A}$ then $\mathcal{I}'$ is also a model of $\mathcal{A}$. Suppose $\mathcal{I}$ is an model of $\mathcal{A}$. Let φ be an assertion from $\mathcal{A}$. We need to show that $\mathcal{I}' \models \varphi$.

- Case $\varphi = (a = b)$: Since $\mathcal{I} \models \varphi$, we have that $a^{\mathcal{I}} = b^{\mathcal{I}}$. By (1), $Z(a^{\mathcal{I}}, a^{\mathcal{I}'})$ and $Z(b^{\mathcal{I}}, b^{\mathcal{I}'})$ hold. Since $a^{\mathcal{I}} = b^{\mathcal{I}}$, by (7), it follows that $a^{\mathcal{I}'} = b^{\mathcal{I}'}$. Hence $\mathcal{I}' \models \varphi$.
- Case $\varphi = (a \neq b)$ is reduced to the above case, by using Lemma 3.1(2).
- Case $\varphi = C(a)$: By (1), $Z(a^{\mathcal{I}}, a^{\mathcal{I}'})$ holds. Since $\mathcal{I} \models \varphi$, $C^{\mathcal{I}}(a^{\mathcal{I}})$ holds. By (13), it follows that $C^{\mathcal{I}'}(a^{\mathcal{I}'})$ holds. Thus $\mathcal{I}' \models \varphi$.
- Case $\varphi = R(a, b)$: By (1), $Z(a^{\mathcal{I}}, a^{\mathcal{I}'})$ holds. Since $\mathcal{I} \models \varphi$, $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$ holds. By (14), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(b^{\mathcal{I}}, y')$ and $R^{\mathcal{I}'}(a^{\mathcal{I}'}, y')$ hold. Consider $C = \{b\}$ (the assumption $O \in \Phi$ is used here). Since $Z(b^{\mathcal{I}}, y')$ and $C^{\mathcal{I}}(b^{\mathcal{I}})$ hold, by (13), $C^{\mathcal{I}'}(y')$ holds, which means $y' = b^{\mathcal{I}'}$. Thus $R^{\mathcal{I}'}(a^{\mathcal{I}'}, b^{\mathcal{I}'})$ holds, i.e., $\mathcal{I}' \models \varphi$.
- Case $\varphi = \neg R(a, b)$ is reduced to the above case, by using Lemma 3.1(2). ◁

Theorem 3.11. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{R}$ be an RBox in $\mathcal{L}_\Phi$. Let $\mathcal{I}_0$ be a model of $\mathcal{R}$, Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and an interpretation $\mathcal{I}_1$, and $\mathcal{I}'_1$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. Then Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}'_1$.*

Proof. We only need to prove that, for every $r \in \Sigma_R$, $x \in \Delta_0^{\mathcal{I}}$, $x', y' \in \Delta^{\mathcal{I}'_1}$:

1. $[Z(x, x') \land r^{\mathcal{I}'_1}(x', y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}_0}[Z(y, y') \land r^{\mathcal{I}_0}(x, y)]$
2. if $I \in \Phi$ then $[Z(x, x') \land r^{\mathcal{I}'_1}(y', x')] \Rightarrow \exists y \in \Delta^{\mathcal{I}_0}[Z(y, y') \land r^{\mathcal{I}_0}(y, x)]$.

We prove these assertions by induction on the timestamps of the steps that extend relations $r^{\mathcal{I}_1}$ to $r^{\mathcal{I}'_1}$, for $r \in \Sigma_R$.

Consider the first assertion. Suppose $Z(x, x')$ and $r^{\mathcal{I}'_1}(x', y')$ hold. We need to show there exists $y \in \Delta^{\mathcal{I}_0}$ such that $Z(y, y')$ and $r^{\mathcal{I}_0}(x, y)$ hold. There are the following three cases:

- Case $r^{\mathcal{I}'_1}(x', y')$ holds because $r^{\mathcal{I}_1}(x', y')$ holds: The assertion holds because Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$.
- Case $r^{\mathcal{I}'_1}(x', y')$ holds because $(\varepsilon \sqsubseteq r) \in \mathcal{R}$ and $y' = x'$: Take $y = x$. Thus, $Z(y, y')$ holds. Since $\mathcal{I}_0$ is a model of $\mathcal{R}$, it validates $\varepsilon \sqsubseteq r$, and hence $r^{\mathcal{I}_0}(x, y)$ also holds.
- Case $r^{\mathcal{I}'_1}(x', y')$ holds because $R_1 \circ \ldots \circ R_k \sqsubseteq r$ is an axiom of $\mathcal{R}$ and there exist $x'_0 = x, x'_1, \ldots, x'_{k-1}, x'_k = y'$ such that $R_i^{\mathcal{I}'_1}(x'_{i-1}, x'_i)$ holds for all $1 \leq i \leq k$: Let $x_0 = x$. For each $1 \leq i \leq k$, since $Z(x_{i-1}, x'_{i-1})$ and $R_i^{\mathcal{I}'_1}(x'_{i-1}, x'_i)$ hold, by the inductive

assumptions of the first two assertions, there exists $x_i \in \Delta^{\mathcal{I}_0}$ such that $Z(x_i, x'_i)$ and $R_i^{\mathcal{I}_0}(x_{i-1}, x_i)$ hold. Thus, $Z(x_k, x'_k)$ holds. Since $\mathcal{I}_0$ validates the axiom $R_1 \circ \ldots \circ R_k \sqsubseteq r$ of $\mathcal{R}$, we also have that $r^{\mathcal{I}_0}(x_0, x_k)$ holds. We choose $y = x_k$ and finish with the proof of the first assertion.

The proof of the second assertion is similar to the proof of the first one. ◁

Theorem 4.1. *Let $\mathcal{I}$ and $\mathcal{I}'$ be finitely branching interpretations (w.r.t. $\mathcal{L}_\Phi$) such that, for every $a \in \Sigma_I$, $a^{\mathcal{I}}$ is $\mathcal{L}_\Phi$-equivalent to $a^{\mathcal{I}'}$. Suppose that if $U \in \Phi$ then $\Sigma_I \neq \emptyset$. Then $x \in \Delta^{\mathcal{I}}$ is $\mathcal{L}_\Phi$-equivalent to $x' \in \Delta^{\mathcal{I}'}$ iff there exists an $\mathcal{L}_\Phi$-bisimulation Z between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x, x')$ holds. In particular, the relation $\{\langle x, x'\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'} \mid x$ is $\mathcal{L}_\Phi$-equivalent to $x'\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

Proof. Consider the "$\Leftarrow$" direction. Suppose Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x, x')$ holds. By (13), for every concept C in $\mathcal{L}_\Phi$, $C^{\mathcal{I}}(x)$ holds iff $C^{\mathcal{I}'}(x')$ holds. Therefore, x is $\mathcal{L}_\Phi$-equivalent to x'.

Now consider the "$\Rightarrow$" direction. Define $Z = \{\langle x, x'\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'} \mid x$ is $\mathcal{L}_\Phi$-equivalent to $x'\}$. We show that Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.

- The assertion (1) follows from the assumption of the theorem.
- Consider the assertion (2) and suppose $Z(x, x')$ holds. By the definitions of Z and $\mathcal{L}_\Phi$-equivalence, it follows that, for every concept name A, $A^{\mathcal{I}}(x)$ holds iff $A^{\mathcal{I}'}(x')$ holds.
- Consider the assertion (3) and suppose that $Z(x, x')$ and $r^{\mathcal{I}}(x, y)$ hold. Let $S = \{y' \mid r^{\mathcal{I}'}(x', y')\}$. We want to show there exists $y' \in S$ such that $Z(y, y')$ holds. Since $x \in (\exists r.\top)^{\mathcal{I}}$ and x is $\mathcal{L}_\Phi$-equivalent to x', we also have that $x' \in (\exists r.\top)^{\mathcal{I}'}$. Hence $S \neq \emptyset$. Since $\mathcal{I}'$ is finitely branching, S must be finite. Let $y'_1, \ldots, y'_n$ be all the elements of S. We have $n \geq 1$. For the sake of contradiction, suppose that, for every $1 \leq i \leq n$, $Z(y, y'_i)$ does not hold, which means y is not $\mathcal{L}_\Phi$-equivalent to y'_i. Thus, for every $1 \leq i \leq n$, there exists a concept C_i such that $y \in C_i^{\mathcal{I}}$ but $y'_i \notin C_i^{\mathcal{I}'}$. Let $C = \exists r.(C_1 \sqcap \ldots \sqcap C_n)$. Thus, $x \in C^{\mathcal{I}}$ but $x' \notin C^{\mathcal{I}'}$, which contradicts the fact that x is $\mathcal{L}_\Phi$-equivalent to x'. Therefore, there exists $y'_i \in S$ such that $Z(y, y'_i)$ holds.
- The assertion (4) can be proved analogously as for (3).
- Consider the assertions (5) and (6) and the case $I \in \Phi$. Observe that the argumentation used for proving (3) are still applicable when replacing r by r^-. Hence the assertion (5) holds. Similarly, the assertion (6) also holds.
- Consider the assertion (7) and the case $O \in \Phi$. Suppose $Z(x, x')$ holds. Take $C = \{a\}$. Since x is $\mathcal{L}_\Phi$-equivalent to x', $x \in C^{\mathcal{I}}$ iff $x' \in C^{\mathcal{I}'}$. Hence, $x = a^{\mathcal{I}}$ iff $x' = a^{\mathcal{I}'}$.
- Consider the assertion (8) and the case $Q \in \Phi$. Suppose $Z(x, x')$ holds. Let $S = \{y \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x, y)\}$ and $S' = \{y' \in \Delta^{\mathcal{I}'} \mid r^{\mathcal{I}'}(x', y')\}$. Since $\mathcal{I}$ and $\mathcal{I}'$ are finitely branching, S and S' must be finite. For the sake of contradiction, suppose there does not exist any bijection $h : S \to S'$ such that $h \subseteq Z$. Thus, there must exist $y'' \in S \cup S'$ such that, for $y_1, \ldots, y_k \in S$ and $y'_1, \ldots, y'_{k'} \in S'$ being all the pairwise different elements of $S \cup S'$ that are $\mathcal{L}_\Phi$-equivalent to y'', we have that $k \neq k'$. Let $\mathcal{I}'' = \mathcal{I}$ if $y'' \in S$, and let $\mathcal{I}'' = \mathcal{I}'$ otherwise. Let $\{z_1, \ldots, z_h\} = S \setminus \{y_1, \ldots, y_k\}$ and $\{z'_1, \ldots, z'_{h'}\} = S' \setminus \{y'_1, \ldots, y'_{k'}\}$. For each $1 \leq i \leq h$, there exists C_i such that $y'' \in C_i^{\mathcal{I}''}$ but $z_i \notin C_i^{\mathcal{I}}$. Similarly, for each $1 \leq i \leq h'$, there exists D_i such that $y'' \in D_i^{\mathcal{I}''}$ but $z'_i \notin D_i^{\mathcal{I}'}$. Let $C = (C_1 \sqcap \ldots \sqcap C_h \sqcap D_1 \sqcap \ldots \sqcap D_{h'})$. We have that $\{y_1, \ldots, y_k\} \subseteq C^{\mathcal{I}}$ and $\{z_1, \ldots, z_h\} \cap C^{\mathcal{I}} = \emptyset$. Similarly, $\{y'_1, \ldots, y'_{k'}\} \subseteq C^{\mathcal{I}'}$ and $\{z'_1, \ldots, z'_{h'}\} \cap C^{\mathcal{I}'} = \emptyset$. If $k > k'$ then $x \in (\geq k\ r.C)^{\mathcal{I}}$ but $x' \notin (\geq k\ r.C)^{\mathcal{I}'}$. If $k < k'$ then $x \notin (\geq k'\ r.C)^{\mathcal{I}}$ but $x' \in (\geq k'\ r.C)^{\mathcal{I}'}$. These contradict the fact that x is $\mathcal{L}_\Phi$-equivalent to x'. Therefore, the assertion (8) must hold.
- Observe that the argumentation used for proving (8) are still applicable when replacing r by r^-. Hence the assertion (9) holds for the case $\{Q, I\} \subseteq \Phi$.

– Consider the assertion (10) and the case $U \in \Phi$. Since $\mathcal{I}$ and $\mathcal{I}'$ are finitely branching and $U \in \Sigma_R$, $\mathcal{I}$ and $\mathcal{I}'$ must be finite. Let $x \in \Delta^{\mathcal{I}}$ and $\Delta^{\mathcal{I}'} = \{x'_1, \ldots, x'_n\}$. For the sake of contradiction suppose that, for all $1 \leq i \leq n$, x'_i is not $\mathcal{L}_\Phi$-equivalent to x. Then, for every $1 \leq i \leq n$, there exists a concept C_i such that $x'_i \in C_i^{\mathcal{I}'}$ but $x \notin C_i^{\mathcal{I}}$. Let $C = (C_1 \sqcup \ldots \sqcup C_n)$ and let a be an individual name. Then $a^{\mathcal{I}'} \in (\forall U.C)^{\mathcal{I}'}$ but $a^{\mathcal{I}} \notin (\forall U.C)^{\mathcal{I}}$, which contradicts the assumption that $a^{\mathcal{I}'}$ is $\mathcal{L}_\Phi$-equivalent to $a^{\mathcal{I}}$. Therefore, there must exist $x'_i \in \Delta^{\mathcal{I}'}$ such that $Z(x, x'_i)$ holds.
– The assertion (11) can be proved analogously as for (10).
– Consider the assertion (12) and the case $\mathtt{Self} \in \Phi$. Suppose $Z(x, x')$ holds. Thus, $x \in (\exists r.\mathtt{Self})^{\mathcal{I}}$ iff $x' \in (\exists r.\mathtt{Self})^{\mathcal{I}'}$. Hence, $r^{\mathcal{I}}(x, x)$ holds iff $r^{\mathcal{I}'}(x', x')$ holds. $\triangleleft$

Theorem 5.2. *For every finitely branching interpretation $\mathcal{I}$, $\equiv_{\Phi,\mathcal{I}}$ is the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$ (i.e. the relations $\equiv_{\Phi,\mathcal{I}}$ and $\sim_{\Phi,\mathcal{I}}$ coincide).*

Proof. If $U \notin \Phi$ or $\Sigma_I \neq \emptyset$ then, by Theorem 4.1, $\equiv_{\Phi,\mathcal{I}}$ is an $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$. Observe that the assumption "if $U \in \Phi$ then $\Sigma_I \neq \emptyset$" is needed for Theorem 4.1 only to prove the assertions (10) and (11). However, such assertions clearly hold for $\mathcal{I}' = \mathcal{I}$ and $Z = \equiv_{\Phi,\mathcal{I}}$. Hence, in the case $U \in \mathcal{L}_\Phi$ and $\Sigma_I = \emptyset$, we also have that $\equiv_{\Phi,\mathcal{I}}$ is an $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$. We now show that it is the largest one. Suppose Z is another $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$. If $Z(x, x')$ holds then, by (13), for every concept C of $\mathcal{L}_\Phi$, $C^{\mathcal{I}}(x)$ holds iff $C^{\mathcal{I}}(x')$ holds, and hence $x \equiv_{\Phi,\mathcal{I}} x'$. Therefore, $Z \subseteq \equiv_{\Phi,\mathcal{I}}$. $\triangleleft$

Theorem 5.3. *If $\Phi \subseteq \{I, O, U\}$ then, for every interpretation $\mathcal{I}$, the relation $Z = \{\langle x, [x]_{\sim_{\Phi,\mathcal{I}}}\rangle \mid x \in \Delta^{\mathcal{I}}\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$.*

Proof. Suppose $\Phi \subseteq \{I, O, U\}$. We have to consider the assertions (1)-(7), (10), (11) for $\mathcal{I}' = \mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$. By the definition of $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$, the assertions (1) and (2) clearly hold. Similarly, the assertion (7) for the case $O \in \Phi$ and the assertions (10), (11) for the case $U \in \Phi$ also hold.

Consider the assertion (3). Suppose $Z(x, x')$ and $r^{\mathcal{I}}(x, y)$ hold. We need to show there exists $y' \in \Delta^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}}$ such that $Z(y, y')$ and $r^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}}(x', y')$ hold. We must have that $x' = [x]_{\sim_{\Phi,\mathcal{I}}}$. Take $y' = [y]_{\sim_{\Phi,\mathcal{I}}}$. Clearly, the goals are satisfied.

For a similar reason, the assertion (5) for the case $I \in \Phi$ holds.

Consider the assertion (4). Suppose $Z(x, x')$ and $r^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}}(x', y')$ hold. We need to show there exists $y \in \Delta^{\mathcal{I}}$ such that $Z(y, y')$ and $r^{\mathcal{I}}(x, y)$ hold. We must have that $x' = [x]_{\sim_{\Phi,\mathcal{I}}}$. Since $r^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}}(x', y')$ holds, there exists $y \in y'$ such that $r^{\mathcal{I}}(x, y)$ holds. Clearly, $y' = [y]_{\sim_{\Phi,\mathcal{I}}}$ and $Z(y, y')$ holds.

For a similar reason, the assertion (6) for the case $I \in \Phi$ holds. $\triangleleft$

Theorem 5.4. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{I}$ be an interpretation. Then:*

1. *For every expression φ which is either a terminological axiom in $\mathcal{L}_\Phi$ or a concept assertion (of the form $C(a)$) in $\mathcal{L}_\Phi$, $\mathcal{I} \models \varphi$ iff $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*
2. *For every expression φ which is either a role inclusion axiom or an individual assertion of the form $R(a, b)$ or $a = b$, if $\mathcal{I} \models \varphi$ then $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*

Proof. The first assertion follows from Theorems 5.3, 3.4 and the definition of $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$. Consider the second assertion. This assertion for the cases when φ is of the form $\varepsilon \sqsubseteq r$, $R(a, b)$ or $a = b$ follows immediately from the definition of $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$. Let $\varphi = (R_1 \circ \ldots \circ R_k \sqsubseteq r)$ and suppose $\mathcal{I} \models \varphi$. We show that $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$. Let $v_0, \ldots, v_k$ be elements of $\Delta^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}}$ such that, for $1 \leq i \leq k$, $R_i^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}}(v_{i-1}, v_i)$ holds. We need to show that $r^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}}(v_0, v_k)$ holds.

For $1 \leq i \leq k$, since $R_i^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(v_{i-1}, v_i)$ holds, there exist $y_{i-1} \in v_{i-1}$ and $z_i \in v_i$ such that $R_i^{\mathcal{I}}(y_{i-1}, z_i)$ holds. Let $x_0 = y_0$. For $1 \leq i \leq k$, since $x_{i-1} \sim_{\Phi,\mathcal{I}} y_{i-1}$ and $R_i^{\mathcal{I}}(y_{i-1}, z_i)$ hold, by (14), there exists x_i such that $x_i \sim_{\Phi,\mathcal{I}} z_i$ and $R_i^{\mathcal{I}}(x_{i-1}, x_i)$ hold, which implies $x_i \in v_i$ and $x_i \sim_{\Phi,\mathcal{I}} y_i$ (when $i < k$). Since $\mathcal{I} \models (R_1 \circ \ldots \circ R_k \sqsubseteq r)$, it follows that $r^{\mathcal{I}}(x_0, x_k)$ holds. Therefore, by definition, $r^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(v_0, v_k)$ holds. $\triangleleft$

Theorem 5.5. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{I}$ be an unreachable-objects-free interpretation. Then:*

1. *$\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is a minimal interpretation $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$.*
2. *If $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finite then it is a minimal interpretation that validates the same terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*
3. *If $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finitely branching then it is a minimal interpretation that satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*

Proof. By Theorem 5.4, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$, validates the same terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$, and satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$.

Clearly, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is an unreachable-objects-free interpretation. Since $\sim_{\Phi,\mathcal{I}}$ is the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$, by Lemma 3.1(4), for $u, v \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$, if $u \neq v$ then u is not $\mathcal{L}_\Phi$-bisimilar to v. Let $Z = \{\langle [x]_{\sim_{\Phi,\mathcal{I}}}, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$. By Theorem 5.3 and Lemma 3.1(2), Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ and $\mathcal{I}$.

Consider the first assertion. Let $\mathcal{I}'$ be any interpretation $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$. We show that $\#\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} \leq \#\Delta^{\mathcal{I}'}$. Let Z' be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$, and let $Z'' = Z \circ Z'$. By Lemma 3.1(3), Z'' is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ and $\mathcal{I}'$. Since $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is an unreachable-objects-free interpretation, by (1), (3) and (5), for every $u \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$, there exists $x_u \in \Delta^{\mathcal{I}'}$ such that $Z''(u, x_u)$ holds. Let $u, v \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ and $u \neq v$. If $x_u = x_v$ then, since u is $\mathcal{L}_\Phi$-bisimilar to x_u and x_v is $\mathcal{L}_\Phi$-bisimilar to v, we would have that u is $\mathcal{L}_\Phi$-bisimilar to v, which is a contradiction. Therefore $x_u \neq x_v$ and we conclude that $\#\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} \leq \#\Delta^{\mathcal{I}'}$.

Consider the second assertion and suppose $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finite. Let $\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = \{v_1, \ldots, v_n\}$. Since $\sim_{\Phi,\mathcal{I}}$ is the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$, by Theorem 4.1 and Lemma 3.1, if $1 \leq i < j \leq n$ then v_i is not $\mathcal{L}_\Phi$-equivalent to v_j. For $1 \leq i, j \leq n$ with $i \neq j$, let $C_{i,j}$ be a concept in $\mathcal{L}_\Phi$ such that $v_i \in C_{i,j}^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ and $v_j \notin C_{i,j}^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$. For $1 \leq i \leq n$, let $C_i = (C_{i,1} \sqcap \ldots \sqcap C_{i,i-1} \sqcap C_{i,i+1} \sqcap \ldots \sqcap C_{i,n})$. We have that $v_i \in C_i^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ and $v_j \notin C_i^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ if $j \neq i$. Let $C = (C_1 \sqcup \ldots \sqcup C_n)$ and, for $1 \leq i \leq n$, let $D_i = (C_1 \sqcup \ldots \sqcup C_{i-1} \sqcup C_{i+1} \sqcup \ldots \sqcup C_n)$. Thus, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ validates $\top \sqsubseteq C$ but does not validate any $\top \sqsubseteq D_i$ for $1 \leq i \leq n$. Any other interpretation with such properties must have at least n elements in the domain. That is, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is a minimal interpretation that validates the same terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.

Consider the third assertion and suppose $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finitely branching. Let $\mathcal{I}'$ be any interpretation that satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$. We show that $\#\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} \leq \#\Delta^{\mathcal{I}'}$. Since Σ_I is finite and $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finitely branching and unreachable-objects-free, $\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ is countable. By Theorem 5.4, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$ and $\mathcal{I}'$. Thus, for every individual name a, $a^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ is $\mathcal{L}_\Phi$-equivalent to $a^{\mathcal{I}'}$. If $\mathcal{I}'$ is not finitely branching then it is infinite and the assertion clearly holds. So, assume that $\mathcal{I}'$ is finitely branching. Then, by Theorem 4.1, $\mathcal{I}'$ is $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$, and hence also $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$. The third assertion thus follows from the first one. $\triangleleft$

Lemma 5.8. *Let $\mathcal{I}$ be a traditional interpretation and let $\mathcal{I}' = \mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$. Then $Z = \{\langle x, [x]_{\sim_{\Phi,\mathcal{I}}}\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}\}$ satisfies all the properties (1)-(7), (10), (11), (13)-(16). In*

particular, the assertion (13) states that, for every concept C in $\mathcal{L}_\Phi$ and every $x \in \Delta^\mathcal{I}$, $x \in C^\mathcal{I}$ iff $[x]_{\sim_{\Phi,\mathcal{I}}} \in C^{\mathcal{I}'}$.

Proof. The properties (1)-(7), (10) and (11) can be shown as in the proof of Theorem 5.3. The properties (13)-(16) can be shown as in Lemma 3.3 except that the case when $Q \in \Phi$ and $C = (\geq n\, R.D)$ and the case when $\mathtt{Self} \in \Phi$ and $C = \exists r.\mathtt{Self}$ for proving the assertion (13) are changed to the following:

- Case $Q \in \Phi$ and $C = (\geq n\, R.D)$, where R is a basic role: Since $Z(x, x')$ holds, we have that $x' = [x]_{\sim_{\Phi,\mathcal{I}}}$. Since $C^\mathcal{I}(x)$ holds, there exist pairwise different $y_1, \ldots, y_n \in \Delta^\mathcal{I}$ such that $R^\mathcal{I}(x, y_i)$ and $D^\mathcal{I}(y_i)$ hold for all $1 \leq i \leq n$. Let the partition of $\{y_1, \ldots, y_n\}$ that corresponds to the equivalence relation $\sim_{\Phi,\mathcal{I}}$ consists of pairwise different blocks $Y_{i_1}, \ldots, Y_{i_k}$, where $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and $y_{i_j} \in Y_{i_j}$ for all $1 \leq j \leq k$. By the inductive assumption, $D^{\mathcal{I}'}([y_{i_j}]_{\sim_{\Phi,\mathcal{I}}})$ holds for all $1 \leq j \leq k$. By the definition of $\mathcal{I}'$, $\mathtt{Q}^{\mathcal{I}'}(R)([x]_{\sim_{\Phi,\mathcal{I}}}, [y_{i_j}]_{\sim_{\Phi,\mathcal{I}}}) = \#Y_{i_j}$ for all $1 \leq j \leq k$. Hence $C^{\mathcal{I}'}([x]_{\sim_{\Phi,\mathcal{I}}})$ holds, which means $C^{\mathcal{I}'}(x')$ holds.
- Case $\mathtt{Self} \in \Phi$ and $C = \exists r.\mathtt{Self}$: Since $Z(x, x')$ holds, we have that $x' = [x]_{\sim_{\Phi,\mathcal{I}}}$. Since $C^\mathcal{I}(x)$ holds, we have that $r^\mathcal{I}(x, x)$ holds. Hence $[x]_{\sim_{\Phi,\mathcal{I}}} \in \mathtt{S}^{\mathcal{I}'}(r)$ and consequently $[x]_{\sim_{\Phi,\mathcal{I}}} \in (\exists r.\mathtt{Self})^{\mathcal{I}'}$, which means $C^{\mathcal{I}'}(x')$ holds. ◁

Theorem 5.9. *Let $\mathcal{I}$ be a traditional interpretation. Then:*

1. *For every expression φ which is either a terminological axiom in $\mathcal{L}_\Phi$ or a concept assertion (of the form $C(a)$) in $\mathcal{L}_\Phi$, $\mathcal{I} \models \varphi$ iff $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*
2. *For every expression φ which is either a role inclusion axiom or an individual assertion of the form $R(a, b)$ or $a = b$, if $\mathcal{I} \models \varphi$ then $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*

Proof. Denote $\mathcal{I}' = \mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ and let $Z = \{\langle x, [x]_{\sim_{\Phi,\mathcal{I}}}\rangle \in \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'}\}$. By Lemma 5.8, for every concept C in $\mathcal{L}_\Phi$, $x \in C^\mathcal{I}$ iff $[x]_{\sim_{\Phi,\mathcal{I}}} \in C^{\mathcal{I}'}$. The first assertion follows immediately from this property. The second assertion can be proved as for Theorem 5.4. ◁

Theorem 5.10. *Let $\mathcal{I}$ be a traditional interpretation without unreachable objects. Then:*

1. *If $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ is finite then it is a minimal QS-interpretation that validates the same terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*
2. *If $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ is finitely branching then it is a minimal QS-interpretation that satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*

Proof. By Lemma 5.8, every $x \in \Delta^\mathcal{I}$ is $\mathcal{L}_\Phi$-equivalent to $[x]_{\sim_{\Phi,\mathcal{I}}}$. Since $\equiv_{\Phi,\mathcal{I}}$ and $\sim_{\Phi,\mathcal{I}}$ coincide, if $[x]_{\sim_{\Phi,\mathcal{I}}} \neq [x']_{\sim_{\Phi,\mathcal{I}}}$ then $[x]_{\sim_{\Phi,\mathcal{I}}}$ and $[x']_{\sim_{\Phi,\mathcal{I}}}$ are not $\mathcal{L}_\Phi$-equivalent to each other. Denote $\mathcal{I}' = \mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$.

Consider the first assertion and suppose $\mathcal{I}'$ is finite. Let $\Delta^{\mathcal{I}'} = \{v_1, \ldots, v_n\}$, where $v_1, \ldots, v_n$ are pairwise different and each v_i is some $[x_i]_{\sim_{\Phi,\mathcal{I}}}$. For $1 \leq i, j \leq n$ with $i \neq j$, let $C_{i,j}$ be a concept in $\mathcal{L}_\Phi$ such that $v_i \in C_{i,j}^{\mathcal{I}'}$ and $v_j \notin C_{i,j}^{\mathcal{I}'}$. For $1 \leq i \leq n$, let $C_i = (C_{i,1} \sqcap \ldots \sqcap C_{i,i-1} \sqcap C_{i,i+1} \sqcap \ldots \sqcap C_{i,n})$. We have that $v_i \in C_i^{\mathcal{I}'}$ and $v_j \notin C_i^{\mathcal{I}'}$ if $j \neq i$. Let $C = (C_1 \sqcup \ldots \sqcup C_n)$ and, for $1 \leq i \leq n$, let $D_i = (C_1 \sqcup \ldots \sqcup C_{i-1} \sqcup C_{i+1} \sqcup \ldots \sqcup C_n)$. Thus, $\mathcal{I}'$ validates $\top \sqsubseteq C$ but does not validate any $\top \sqsubseteq D_i$ for $1 \leq i \leq n$. Any other QS-interpretation with such properties must have at least n elements in the domain. That is, $\mathcal{I}'$ is a minimal QS-interpretation that validates the same terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.

Consider the second assertion and suppose $\mathcal{I}'$ is finitely branching. Let $\mathcal{I}''$ be any other QS-interpretation that satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$. We show that

$\#\Delta^{\mathcal{I}'} \leq \#\Delta^{\mathcal{I}''}$. Since Σ_I is finite and $\mathcal{I}'$ is finitely branching and unreachable-objects-free, $\Delta^{\mathcal{I}'}$ is countable. If $\mathcal{I}''$ is not finitely branching then it is infinite and the assertion clearly holds. So, assume that $\mathcal{I}''$ is finitely branching. Since $\mathcal{I}$ is unreachable-objects-free and $\mathcal{I}''$ is a finitely branching QS-interpretation that satisfies the same concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$, it can be shown that, for every $x \in \Delta^{\mathcal{I}}$, there exists $x'' \in \Delta^{\mathcal{I}''}$ that is $\mathcal{L}_\Phi$-equivalent to x. Recall that every $x' \in \Delta^{\mathcal{I}'}$ is $\mathcal{L}_\Phi$-equivalent to some $x \in \Delta^{\mathcal{I}}$, and if x'_1 and x'_2 are different elements of $\Delta^{\mathcal{I}'}$ then they are not $\mathcal{L}_\Phi$-equivalent to each other. This implies that $\#\Delta^{\mathcal{I}'} \leq \#\Delta^{\mathcal{I}''}$. ◁

Theorem 6.1. *Algorithm 1 is correct and can be implemented to have time complexity* $O(|\Sigma| \cdot n \cdot \log n)$*, where* $n = |\Delta^{\mathcal{I}}|$.

Proof. (Sketch) It is straightforward to prove correctness of Algorithm 1.

To estimate complexity, note that the steps 4-14 of Algorithm 1 are essentially the same as the fragment of Hopcroft's automaton minimization algorithm [10,18] used for refining the partition. The only difference occurs when $Q \in \Phi$ and splitting X by a pair $\langle Y, R\rangle$ may result in more than two subblocks. The way for dealing with that case has been mentioned earlier. The complexity analysis of [18] can be applied to Algorithm 1 without any essential changes, and we conclude that the steps 4-14 can be implemented to have time complexity $O(|\Sigma_R^\dagger| \cdot n \cdot \log n)$ (i.e. $O(|\Sigma_R| \cdot n \cdot \log n)$).

Consider complexity of the step 2 of Algorithm 1. To compute abstract classes of the equivalence relation $ECond_\Phi$, we start from the partition $\{\Delta^{\mathcal{I}}\}$ and then:

1. Refine the current partition by using the condition that, when $O \in \Phi$, x and x' should be in the same block only if, for every $a \in \Sigma_I$, $x = a^{\mathcal{I}}$ iff $x' = a^{\mathcal{I}}$. This can be done in $O(|\Sigma_I|)$ steps.
2. Refine the current partition by using the condition that, when $\mathtt{Self} \in \Phi$, x and x' should be in the same block only if, for every $r \in \Sigma_R$, $\langle x, x\rangle \in r^{\mathcal{I}}$ iff $\langle x', x'\rangle \in r^{\mathcal{I}}$. This can be done in $O(|\Sigma_R| \cdot n)$ steps.
3. Refine the current partition by using the condition that x and x' should be in the same block only if, for every $A \in \Sigma_C$, $x \in A^{\mathcal{I}}$ iff $x' \in A^{\mathcal{I}}$. We do this as follows: first, for each x in a non-singleton block, we sort the list $\{A \in \Sigma_C \mid x \in A^{\mathcal{I}}\}$ by using "counting sort" (the total time for this is $O(|\Sigma_C| \cdot n)$); next, for each block of the current partition, we sort elements of the block by those lists, and then group elements that have the same list of that kind into a subblock, which forms a new block for the partition. The total time is $O(|\Sigma_C| \cdot n + n \cdot \log n)$.

Summing up, the time complexity of the step 2 of Algorithm 1 is of rank $O(|\Sigma_I| + |\Sigma_R| \cdot n + |\Sigma_C| \cdot n + n \cdot \log n)$. Therefore, Algorithm 1 can be implemented to have time complexity $O(|\Sigma| \cdot n \cdot \log n)$. ◁